A priori bounds on legislative bargaining agreements *

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Abstract

In a workhorse model of legislative bargaining with spatial preferences, I establish easy to compute inner and outer bounds on all equilibrium acceptable agreements and outcomes. Using these bounds, I recover and extend conditions for uniqueness of equilibrium, and derive novel proposer advocacy conditions for outcomes to be near the core or the uncovered set. The centrifugal effect of these conditions is reversed as proposal rights are concentrated further away from the center. In the discounted case, there is an outer limit to proposal influence, which becomes non-binding as legislators get patient. The location of the status quo has the expected effect, while the effect of changes in the voting rule is ambiguous. While providing a practical avenue to locating legislative decisions, the analysis highlights the broad conclusion that the proper functioning of democratic institutions is highly contingent on other institutional features besides the proper assignment of voting rights.

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What policies do legislatures, parliaments, or committees adopt when deciding over a continuum of alternatives? How do these policy outcomes change as the voting rule, rules for the origination of proposals, and other features of the legislative environment vary? Even the first of these questions has no obvious answer. Plott (1967) convincingly showed that a natural candidate set of collective outcomes in these settings, the core, exists only under knife-edge conditions when the policy space comprises more than one dimensions. Richard McKelvey and Norman Schofield (McKelvey (1976, 1979); Schofield (1978)) subsequently showed that in the generic case when such knife-edge conditions are not met, intransitivity of majority preferences afflicts the committee in the most egregious manner, encompassing the entire policy space. These findings posed both a theoretical and a practical challenge for the study of policy-making in committees. On the theoretical side, they forcefully introduced the possibility that preference aggregation in legislatures may generate outcomes that poorly reflect collective preferences and may be subject to normatively unacceptable levels of manipulation by a minority of legislators with procedural prerogatives (McKelvey (1979); Riker (1980)). On the applied side, they left researchers with little guidance as to what to substitute for core policies when specifying likely legislative policy outcomes. Several generalizations of the core were subsequently developed to address the theoretical and practical gap in the literature.\(^1\)

With the subsequent shift in the literature from a cooperative to a non-cooperative game-theoretic approach, the workhorse model to address these questions has been the legislative bargaining model of Baron and Ferejohn (1989). However, most applications using this framework primarily operate in a divide-the-dollar setting (for example, see the recent review by Eraslan and Evdokimov (2019)). Indeed, while the model can certainly accommodate more general spaces of political disagreement (Banks and Duggan (2000, 2006)),

\(^1\)See Austen-Smith and Banks (1999) for a review.
it is rarely analytically tractable with spatial preferences except in special environments.\textsuperscript{2} The goal of this study is to develop a computationally feasible method to delineate possible outcomes in the legislative bargaining model with classic spatial preferences, thus providing an inexpensive alternative to the computation of equilibrium for practitioners and, at the same time, allowing a systematic theoretical exploration of the dependence of collective decisions on the allocation of proposal and voting rights and other features of the legislative environment.

I study committees that decide by general voting rules and bargaining protocols, and I allow for large committees, including the possibility of a continuum of members or a separate (possibly continuous) population of proposers. The model captures weighted majority quota rules, multi-cameral legislatures, and veto and veto-override provisions as provided by the US constitution and, in the typical application with finite committees, it encompasses all monotonic simple voting rules. Though some findings apply to a more general domain of preferences, I assume negative quadratic Euclidean preferences, a form prevalent in the literature both in applications of the model but also in empirical work to estimate legislator preferences. For any point $y$, I establish inner and outer spherical bounds centered at $y$ on acceptable agreements such that no proposal outside the outer sphere is socially acceptable in any equilibrium, and all proposals within the (possibly empty) inner sphere are socially acceptable in every equilibrium (Theorem 2). Figure 1 illustrates. These bounds on socially acceptable outcomes also directly imply (more informative) bounds on possible decisions the committee may reach in any equilibrium. Importantly, these a priori restrictions on equilibrium outcomes can be derived easily using straightforward computations: the radii of the bounds $\bar{B}^*(y), \bar{B}^*(y)$, each solve one equation in one unknown, and can be obtained by elementary numerical methods such as bisection or are otherwise available in closed form (Theorem 3). These bounds can become sharper by judicious choice of the center point $y$.

\textsuperscript{2}Primarily one-dimensional instances of the spatial model (e.g., Cardona and Ponsati (2011); Cho and Duggan (2003); Herings and Predtetchinski (2010); Predtetchinski (2011)) or a small number of players in higher dimensions (e.g., Baron (1991)).
For all $y$, there exists a ball with radius $\bar{B}^*(y)$ that contains all socially acceptable agreements in every equilibrium. An open ball with radius $\bar{B}^*(y)$ (possibly zero) contains agreements that are acceptable in every equilibrium. $\bar{B}^*(y)$, $\bar{B}^*(y)$ each are solutions of one equation in one unknown.

or by combining bounds with different centers and I highlight guidelines to that effect in Corollary 2.

Before I summarize the comparative statics questions these bounds help address, I now briefly outline how they are established. The analysis builds on two generalizations of the median: A $y$-centered $P$-ball (for pivotal) is a smallest radius ball such that if all of its members prefer one alternative over another, then a winning coalition also prefer that alternative. A $y$-centered $W$-ball (for winning) is a smallest radius ball such that if a winning coalition prefer an alternative over another then there exists some member of the $y$-centered $W$-ball that also prefers that alternative. Using these generalized medians, I recover a structure of the set of socially acceptable agreements (Lemma 2) comparing individual payoffs to possible lotteries over agreements induced in any equilibrium. That structure, in turn, allows me to bound the maximum and minimum distance of optimal individual proposals from $y$ as a function of equilibrium-specific bounds centered at $y$ (Lemma 3). These arguments yield inequations for equilibrium-specific bounds and in a final step I rely once more on the generalized medians to relax these inequalities so that they apply to bounds across equilibria.
(Theorem 1). The resulting equilibrium bounds centered at any $y$ depend indirectly on the voting rule through the radii of the corresponding $P$- and $W$-balls, and directly on other model primitives such as the status quo, discount factor, and the distribution of proposers. The radii of $y$-centered $P$- and $W$-balls are easy to compute. Because the bounds may generally be sharper when centered at points $y$ with minimum $P$- or $W$-ball radius, the extra computation cost to locate such centers is likely warranted.\footnote{When the radius is minimized over possible centers $y$, these constructs bear a close connection to the classic majority-rule yolk (Ferejohn, McKelvey and Packel (1984); McKelvey (1986)), with which they coincide in a unicameral legislature with an odd number of committee members and simple majority rule. With general voting rules, $P$-balls and $W$-balls are generally distinct.} Fortunately, that additional computation can also be performed efficiently in finite committees by an adaptation of the algorithm of Craig Tovey (Tovey (1992)), which is polynomial in the size of the committee (for fixed policy dimension).

I establish comparative statics on the equilibrium bounds with respect to the location of the status quo, showing that they expand as it gets further away from the center of the bounds (Theorem 6). In the discounted case, the inner and outer bound differ by a fixed constant at the limit as the distance of the status quo from the center of the bounds goes to infinity, implying that for bad enough status quo a unique equilibrium in pure strategies prevails if the distribution of proposers has bounded support (Corollary 4). I also study the effect of changes in the voting rule and show that more restrictive (proper) voting rules, such as higher majority quota thresholds in one chamber or the addition of a legislative chamber, have ambiguous effects on the outer bound though the inner bound weakly contracts (Theorem 10). On the one hand, such changes tend to shrink the set of socially acceptable outcomes in any equilibrium. A counter-veiling force emerges, though, because more stringent voting rules expand the set of core points (once the core is non-empty) and may also expand the set of possible equilibria.

What do these bounds imply about the possible or likely location of equilibrium
outcomes? First, for any fixed instance of the model, the radius of the outer bound is finite (Theorem 3), therefore social choice induced by the non-cooperative legislative bargaining model cannot wander everywhere in the space of alternatives. This is so if the distribution of proposers has finite second moments\(^4\) and in Online Appendix B I provide an example in which that moment condition is violated and there exists an equilibrium with outcome support on the entire real line. Furthermore, under the assumed moment restriction on the distribution of proposers, finiteness of the bound also relies on the structure of the set of socially acceptable proposals induced by quadratic preferences (Lemma 2) which disciplines the distribution of optimal proposals (Lemma 3).

Second, I explore conditions for equilibrium outcomes to be at or near core points (if they exist) or uncovered points (Fishburn (1977); Miller (1980)). I term these conditions advocacy conditions, as they generally amount to the existence or prevalence of proposers that advocate for such outcomes. The weaker of these advocacy conditions are generalizations of conditions of Banks and Duggan (2000, 2006) for core equivalence (Theorem 4, part 2) and require that a unique core is in the support of the distribution of proposers when legislators are perfectly patient. In the general – discounted or not – case, Theorem 5 establishes a lower bound on the probability equilibrium outcomes fall within an envelope containing the core or the uncovered set, where the bounding probability is equal to the probability of proposers sufficiently close to the center of that envelope.

Thirdly, I show that the outer bound expands as the distribution of proposers piles mass away from its center (Theorem 8, part 1). In the discounted case, I show that there is a maximal bound that limits the effect of changes in the distribution of proposers for any fixed discount factor. The radius of this maximal bound coincides with the distance of the center of the bound from the status quo if the bound is centered at a core point, but if the core is empty it increases with the discount factor and, contrary to the expectation that equilibrium outcomes are confined in the center of the policy space, it encompasses the entire policy space at the limit as the discount factor goes to one (Theorem 8, parts

\(^4\)This restriction is automatically met if the distribution of proposers has finite support.
2-3). Furthermore, if the core is empty, then for any arbitrary distance from any point $y$ and sufficiently patient legislators, there exists a distribution of proposers that guarantees the bulk of equilibrium outcomes are farther than that distance (Theorem 8, part 4). In the limit model with perfectly patient legislators, I show that if there exists a unique proposer, then the radius of the outer bound is larger than the distance of that proposer’s ideal from the center of the bound, no matter how far that ideal is. Under these assumptions (patient players and a single proposer), I also show there exists an equilibrium in which the single proposer can pass her ideal point with probability one if and only if there does not exist a legislator with absolute veto who prefers the status quo more than the ideal point of the proposer (Theorem 9).

In sum, in the spatial legislative bargaining model proposal rights have a bounded effect on equilibrium outcomes for any fixed discount factor less than one, and may support the core or its generalizations if they satisfy the advocacy conditions of Theorems 4 or 5. But without a core, manipulation of the rights for the origination of proposals can nullify these centrifugal tendencies as the legislators become patient. Notably, patience does not deliver this result through complex history-dependent equilibrium constructions, and the analysis throughout focuses on equilibria in stationary strategies. Nor is it necessary to manipulate proposal rights using non-stationary recognition rules. These latter findings parallel and complement a number of recent studies on the role of proposal rights in collective decision-making. In the divide-the-dollar setting, Ali, Bernheim and Fan (2019) expand the set of recognition protocols to non-stationary rules and study the role of predictability in allowing a proposer to extract the whole dollar. Kalandrakis (2006) shows that even with stationary recognition protocols all possible equilibrium expected payoffs are attainable for some allocation of proposal rights. A number of papers focus on the special case of a persistent proposer (as in Theorem 9) in a similar class of bargaining models including Primo (2002) and Duggan and Ma (2018). Kalandrakis (2010), Diermeier and Fong (2011), Diermeier and Fong (2012), and Anesi and Duggan (2016) study the special case of a persistent proposer in sequential bargaining models with endogenous status quo.
The reliance of the approach pursued in this paper on generalized medians and their connection to the yolk directly links this work with an earlier wave of the literature on collective decision-making motivated by the same broad questions that motivate this study. The yolk first appears in the work of Ferejohn, McKelvey and Packel (1984) who study a dynamic non-equilibrium process of agenda formation that induces a Markov process over policy outcomes. McKelvey (1986) used the yolk in an incisive construction of an outer bound for the uncovered set in the classic spatial model. McKelvey and several authors established a connection between the uncovered set and Downsian electoral competition (e.g., Banks, Duggan and LeBreton (2002); Dutta and Laslier (1999); McKelvey (1986)), or certain legislative amendment procedures (McKelvey (1986); Shepsle and Weingast (1984)). Gary Cox (Cox (1987)) generalized McKelvey’s conclusion on the central location of the uncovered set to non-Euclidean preferences. Feld, Grofman and Miller (1988) developed a priori bounds on the size of the yolk; while Feld, Grofman and Miller (1989) suggest that the ability to manipulate the agenda in committees may be limited when the yolk is small. All of these studies have as common theme a direct or implied restriction on likely social choice outcomes to a centrally located set of the policy space that shrinks as the committee preferences get closer to satisfying conditions for the existence of a core point. As already discussed, there are similarities and important differences between the main thrust of the conclusions in these studies and the information that can be gleaned from the equilibrium bounds established presently.

I conclude this section with some comments on computational aspects of the analysis. An important advantage of the approach of this paper is that it obtains bounds on all equilibria at an overall cost that is polynomial in the size of the committee. Therefore, this approach is generally less expensive than computation of even a single equilibrium. Though there are versions of the legislative bargaining model (in the divide-the-dollar case) for which a polynomial time algorithm to compute equilibrium expected payoffs exists (Kalandrakis...
no such algorithm is known to exist for the model with spatial preferences. The bounds proposed presently achieve a certain efficiency, in that their computation does amount to computation of the unique equilibrium in special cases when equilibrium is unique. In fact, I show that the bounds recover and extend the uniqueness result of Cho and Duggan (2003) to multiple dimensions and infinite committees (Theorem 7). But, in general, computing one equilibrium of this model amounts to computing a Brouwer fixed point, which is a computational problem that belongs in the class PPAD (Papadimitriou (1994)) and is considered a hard problem. Furthermore, the proposed approach bounds all equilibria, whilst finding all Brouwer fixed points is an even harder computational problem (Herings and Peeters (2010); McKelvey and McLennan (1996)). Finally, even if an actual equilibrium is desired, the bounds provide a natural starting point to economize on its computation.

1 Model

Consider policy-making over a $D$-dimensional policy space, $X = \mathbb{R}^D$, where $D \geq 1$. To allow large committees with a continuum of members let the set of players be $I = \mathbb{R}^D$. Players are indexed by $\hat{x} \in I$, and are endowed with a von Neumann-Morgenstern utility function $u : X \times I \rightarrow \mathbb{R}$ over policies, $x$, that takes the familiar negative quadratic form $u(x; \hat{x}) = -(x - \hat{x})^T (x - \hat{x})$. Let $\pi$ be a probability measure on $I$, and assume that it has finite first and second moments. In each period $t = 1, 2, \ldots$ before an agreement is reached, a proposer $\hat{x}$ is realized from distribution $\pi$, independently across periods. The proposer offers a proposal $z$ that is put to an up-or-down vote and if the set of legislators $A \subseteq I$ that

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5One important difference between these models is that in the divide-the-dollar framework, equilibrium expected payoffs are known to be unique (Eraslan (2002); Eraslan and McLennan (2013)), but payoff uniqueness does not generally hold in the spatial model.

6Not only do the bounds provide an informative starting point to build an equilibrium set of socially acceptable agreements, but they also fully determine pure proposal strategies for the subset of players with ideal points within the inner bound (Lemma 3).
approve it satisfies (1) (as explained below), then \( z \) is implemented in all remaining periods and the game ends; otherwise, a status quo policy \( q \in X \) is implemented in period \( t \) and the game moves to period \( t+1 \) with a new proposer drawn from \( \pi \), a new proposal, etc., until an agreement is reached. Players discount the future by a common factor \( \delta \in [0, 1] \) and player \( \hat{x} \)'s payoff if agreement \( x \) is reached in period \( t \) is given by

\[
(1 - \delta^{t-1})u(q; \hat{x}) + \delta^{t-1}u(x; \hat{x})
\]

(and it is \( u(q; \hat{x}) \) in the case of perpetual disagreement).

Voting over policies is organized in a finite number of \( L, L \geq 1 \), legislative chambers indexed by \( \ell = 1, \ldots, L \), each represented by a pair \( \mu_\ell, m_\ell \): \( \mu_\ell \) is a probability measure over \( \mathcal{I} \) which can be construed as the distribution of voting weights in chamber \( \ell \); \( m_\ell \in (0, 1) \) is a threshold of support that is required in chamber \( \ell \) in order for a policy to be approved in that chamber. Proposal \( x \) is approved if the set of legislators \( A \subseteq \mathcal{I} \) that vote for it are a winning coalition in each chamber

\[
(1) \quad \mu_\ell(A) \geq m_\ell, \text{ for all } \ell = 1, \ldots, L.
\]

A single legislature deciding by majority rule is the special case when \( L = 1 \) and \( m_1 = \frac{1}{2} \).

The equilibrium concept, which is standard in this literature, is subgame perfect Nash in stationary, no-delay,\(^7\) proposal strategies, and voting strategies that are deferential or stage-undominated (Baron and Kalai (1993)). To formalize, let \( \mathcal{P} \) denote the space of Borel probability measures on \( X \). A strategy profile is a pair consisting of a measurable proposal profile \( p: \mathcal{I} \to \mathcal{P} \) and a voting profile \( A: \mathcal{I} \to 2^X \). Here \( p_{\hat{x}} \in \mathcal{P} \) is the probability distribution over proposals when the proposer has ideal point \( \hat{x} \) and \( A(\hat{x}) \subseteq X \) is the set

\(^7\)It is well understood that the restriction to no-delay proposal strategies is without loss of generality in this model. Delay is only possible in knife-edge situations (see Banks and Duggan (2006)), and in the non-generic cases such equilibria with delay prevail, they are outcome equivalent to an equilibrium without delay. For the exact same reasons, the restriction to deferential voting strategies is of no consequence under the maintained assumption on payoffs.
of proposals that committee member with ideal \( \hat{x} \) is approving. Let \( \sigma = (A, p) \) denote a complete such strategy profile. Given voting strategy profile \( A \) the set of socially-acceptable agreements is given (according to (1)) by \( A_\sigma = \{ x \in X \mid \mu_\ell (\{ \hat{x} \mid x \in A(\hat{x}) \}) \geq m_\ell \text{ for all } \ell \} \).

With proposal strategies that are no-delay,\(^8\) that is, that place probability one on \( A_\sigma \), the continuation payoff of \( \hat{x} \) is given by

\[
v_\sigma(\hat{x}) = (1 - \delta) u(q; \hat{x}) + \delta \int \int u(z; \hat{x}) p_{\hat{x}'}(dz) \pi(d\hat{x}') .
\]

Two conditions are necessary for such \( \sigma = (A, p) \) to be an equilibrium:

\[(E_v) \quad A(\hat{x}) = \{ x \mid u(x; \hat{x}) \geq v_\sigma(\hat{x}) \},\]

\[(E_p) \quad p_{\hat{x}} \left( \arg \max_x \{ u(x; \hat{x}) \mid x \in A_\sigma \} \right) = 1 .\]

While these conditions preclude profitable one-stage deviations, additional conditions must be verified in general to preclude profitable infinite deviations.\(^9\) I will refer to a profile \( \sigma \) that satisfies \((E_v)\) as a voting equilibrium. I call \( \sigma \) a quasi-equilibrium if it also satisfies \((E_p)\), to emphasize that profitable infinite deviations have not been ruled out (yet). I call a profile \( \sigma \) an equilibrium when profitable infinite deviations have also been precluded.

A few comments are in order regarding this model setup. First, allowing for a continuum (or even countably infinite) number of players is a significant departure from existing literature. Infinite committees conveniently capture direct democracy settings in which proposals are approved through referendum in a large electorate that can be well-approximated by a continuum. Furthermore, the support of probability measures \( \mu_1, \ldots, \mu_L, \pi \) is unrestricted. It is possible, for example, that the distribution of proposers has full support so that all conceivable positions might be advocated, albeit with vanishing likelihood, in society. Finitely populated legislatures are special cases when each \( \mu_\ell \) is a discrete measure with finite support. The set of proposers can be restricted among players with voting rights

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\(^8\)See footnote 7.\(^9\)This is so even in the discounted case (\( \delta < 1 \)) because we have allowed payoffs to be unbounded. The bounds established in Section 2 restore that property and every quasi-equilibrium is indeed an equilibrium in the discounted case (Corollary 3).
(\text{Support}(\pi) \subseteq \bigcup_{\ell=1}^{L} \text{Support}(\mu_\ell))$ or we can assume proposals to the legislature arise from a continuum of outside members such as an electorate, lobbies, interest groups, etc. The only restriction placed on the distribution of proposers, $\pi$, is that it has finite first and second moments, a restriction that is automatically met in the finite committee setting.

It is not consequential that we require legislators to have distinct ideal points. We can equivalently capture the case two or more legislators share an ideal point $\hat{x}$ by accordingly increasing the voting weight (and possibly proposal probability) of $\hat{x}$. Under this interpretation, legislators that share an ideal point are restricted to play the same strategy. With regard to voting strategies, this restriction is already imposed broadly in the literature via condition $(E_v)$. The restriction on proposal strategies also does not affect the equilibrium outcome distribution: Every equilibrium such that two or more legislators with identical ideal points employ distinct proposal strategies can be transformed to an equivalent equilibrium in which they share the same proposal strategy. Finally, the restrictions on the voting rule are also minimal. As shown by Taylor and Zwicker (1993) any multi-cameral legislature, along with veto and veto-override provisions similar to those found in the US constitution, are easily accommodated. In fact, all monotonic voting rules can be captured if the committee is finite (Taylor and Zwicker (1993)).

With these comments on the model, we can proceed to the analysis. Before doing so, I introduce additional necessary notation. Let a hyperplane $H_{a,c}$ in $\mathbb{R}^D$ be defined by direction $a \in \mathcal{A}$ and level $c$ such that $H_{a,c} = \{x \mid a^T \cdot x = c\}$, where $\mathcal{A} = \{a \in \mathbb{R}^D \mid ||a|| = 1\}$ is the set of unit vectors in $\mathbb{R}^D$. Let $H_{a,c}^+ = \{x \mid a^T \cdot x \geq c\}$ and $H_{a,c}^- = \{x \mid a^T \cdot x \leq c\}$ be the two closed half-spaces defined by $H_{a,c}$. If the direction $a$ and level $c$ are not needed in the context, I denote a hyperplane by $H$, and let $H^+$ and $H^-$ be the corresponding (arbitrary) two half-spaces. For any $x \in \mathbb{R}^D$, let $\mathcal{B}(x,d) = \{x' \in \mathbb{R}^D \mid d(x,x') \leq d\}$ be the closed ball of $\mathbb{R}^D$.

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10With $\mu_\ell$ then being counting measures. Note that the restriction that $m_\ell \in (0,1)$ is not consequential in this finite setting. In particular, unlike in the infinite committee setting, $m_\ell < 1$ does not preclude the unanimity rule because we can raise $m_\ell$ above the sum of voting weights of all except the committee member with the smallest voting weight.
points within distance $d$ from $x$, and denote the corresponding open ball by $B^o(x, d)$.

2 Equilibrium bounds

To illustrate the approach, I will first show its incarnation in the special case of the model when possible agreements fall in one dimension ($D = 1$), there is one chamber ($L = 1$) operating under simple majority rule ($m_1 = \frac{1}{2}$), and there is a unique median committee member at $\hat{x}_m \in \mathbb{R}$. In this setting there is a unique equilibrium when $\delta < 1$ and the committee is finite (see Cho and Duggan (2003)). Uniqueness of equilibrium aside, for any equilibrium $\sigma$ the majority-acceptable agreements are given by an interval $A_\sigma = [\hat{x}_m - B_\sigma, \hat{x}_m + B_\sigma]$ for some scalar $B_\sigma \geq 0$. It follows that the distance of $\hat{x}$’s equilibrium proposal from the median is

\begin{equation}
\tilde{d}(B_\sigma, \hat{x}) = \min\{d(\hat{x}_m, \hat{x}), B_\sigma\}.
\end{equation}

This allows us to express the continuation payoff of the median as

$$v_\sigma(\hat{x}_m) = -\left((1 - \delta)(d(\hat{x}_m, q))^2 + \delta \int(\tilde{d}(B_\sigma, \hat{x}))^2 \pi(d\hat{x})\right),$$

and pin down the distance, $B_\sigma$, of the farthest alternative from $\hat{x}_m$ the median approves $u(\hat{x}_m + B_\sigma; \hat{x}_m) = u(\hat{x}_m - B_\sigma; \hat{x}_m) = v_\sigma(\hat{x}_m)$. Therefore, $B_\sigma$ is a solution for $B$ to

\begin{equation}
B^2 - \left((1 - \delta)d(\hat{x}_m, q))^2 + \delta \int(\tilde{d}(B, \hat{x}_m))^2 \pi(d\hat{x})\right) = 0.
\end{equation}

Generically, equation (3) admits a unique solution, but to cover cases with multiple equilibria, define $B^*$ to be its largest solution. This coincides with the general outer bound $\bar{B}^*(\hat{x}_m)$ I characterize later in this section,\textsuperscript{11} that is, in every equilibrium $\sigma$, possible equilibrium agreements are confined within $B(\hat{x}_m, B^*) \supseteq A_\sigma$. What is particularly appealing in this

\textsuperscript{11}If there is a unique equilibrium, $B^*$ also coincides with the inner bound $B^*(\hat{x}_m)$.
special setting is that the computation involved could not be simpler: $B^*$ solves one equation in one unknown.

Two stumbling blocks stand in the way of obtaining an equation analogous to (3) more generally. First, the set of socially-acceptable agreements does not coincide with the set of agreements acceptable by some decisive player, such as a unique median, whose preferences can be used to determine the bound on equilibrium agreements as in equation (3). In more than one dimensions, such decisive player does not exist generally or even generically. Second, when $D > 1$ the optimal proposals cannot be determined \textit{a priori} as a function of the bound in the sense of equation (2) since the socially-acceptable set $A_\sigma$ is generally not convex and has no standard geometric form. This is so even if equilibrium is unique and in pure strategies, though in more than one dimensions equilibrium need not be unique and may require non-degenerate mixed proposal strategies.

To overcome the first obstacle, I use two generalizations of the median introduced in a companion working paper, but I introduce these concepts here for completeness and to make the analysis self-contained. Define a \textit{pivotal} hyperplane as one that renders one of its half-spaces a smallest such (that is, half-space) winning coalition. Specifically:

\textbf{Definition 1.} Hyperplane $H_{a,c}$ is pivotal if $c = c^*(a)$ where

$$c^*(a) = \max_c \{ c \mid \mu_\ell(\{ \hat{x} \mid a^T \cdot \hat{x} \geq c \}) \geq m_\ell \text{ for all } \ell \}.$$

For every $a \in \mathcal{A}$, $H_{a,c^*(a)}^+$ is a pivotal winning half-space.

We can now define the two median generalizations for an arbitrary point $y$, and subsequently refine those definitions to determine recommended center points for each bound. A $y$-centered $P$-ball (for pivotal) is a smallest radius ball $B(y,r)$ that intersects all pivotal hyperplanes, that is, $r$ solves

$$(P_y) \quad \min_r \{ r \mid B(y,r) \cap H_{a,c^*(a)} \neq \emptyset \text{ for all } a \in \mathcal{A} \}.$$
Analogously define for all \( y \) a \( y \)-centered \( W \)-ball (for winning) as a smallest radius ball \( B(y, r) \) that intersects all pivotal winning half-spaces

\[
(W_y) \quad \min_r \{ r \mid B(y, r) \cap H_{a,c^*(a)}^+ \neq \emptyset \text{ for all } a \in A \}.
\]

Denote the minimizer (and attained minimum) of programs \((P_y)\) and \((W_y)\) by \( r_p(y) \) and \( r_w(y) \). It is easy to see that \( r_p(y) \geq r_w(y) \) for all \( y \). These constructs allow us to build equilibrium bounds centered at arbitrary \( y \), but such bounds are generally sharper when the corresponding radii \( r_p(y), r_w(y) \) are minimized. Thus, searching over both a center \( y \) and radius \( r \), a \( P \)-ball solves

\[
(P) \quad \min_{y,r} \{ r \mid B(y, r) \cap H_{a,c^*(a)}^+ \neq \emptyset \text{ for all } a \in A \}.
\]

while a \( W \)-ball solves

\[
(W) \quad \min_{y,r} \{ r \mid B(y, r) \cap H_{a,c^*(a)}^+ \neq \emptyset \text{ for all } a \in A \}.
\]

Let \( y_p, r_p \) and \( y_w, r_w \) denote solutions of \((P)\) and \((W)\), respectively. These generalized medians have a connection with the classic majority rule yolk (Ferejohn, McKelvey and Packel (1984); McKelvey (1986)): Programs \((P)\) and \((W)\) coincide under single chamber simple majority rule with an odd number of legislators, and their solutions coincide with a classic yolk. It is understood (Martin, Nganmeni and Tovey (2016)) that even the classic yolk need not be unique in general, and such indeterminacy can certainly be expected for \( P \)-balls and \( W \)-balls. I refer to \( y_p \) as a \( P \)-center and \( y_w \) as a \( W \)-center, and in general \( y_p \neq y_w \), even if these solutions are unique.

Lemma 1 renders precise the claim that these constructs function as generalized medians: Any socially-acceptable agreement of a voting equilibrium must be weakly preferred by some player with ideal point in the \( y \)-centered \( W \)-ball; and, for all \( y \), if all \( \hat{x} \) in the \( y \)-
centered $P$-ball weakly prefer $x$ over their equilibrium continuation payoff, then $x$ is socially-acceptable in a voting equilibrium.

**Lemma 1.** Let $\sigma$ be a voting equilibrium. For all $y$ and all $x$

1. If $x \in A_\sigma$, then there exists $\hat{x} \in B(y, r_p(y))$ such that $u(x; \hat{x}) \geq v_\sigma(\hat{x})$.

2. If $u(x; \hat{x}) \geq v_\sigma(\hat{x})$ for all $\hat{x} \in B(y, r_p(y))$, then $x \in A_\sigma$.

Building on Lemma 1, the next Lemma ensures that if a policy $x$ is socially acceptable, then for all $y$, any policy that is closer to $y$ than $x$ by a distance $2(r_p(y) + r_w(y))$ is also acceptable. This result bears a close connection to arguments of McKelvey (1986) though he pursues such arguments only for simple majority rule. Furthermore, in the present setting, the argument applies to arbitrary center points $y$ and proposals are compared to equilibrium continuation lotteries instead of deterministic alternatives. To economize on notation, in what follows I set $r(y) := r_p(y) + r_w(y)$.

**Lemma 2.** For all $y, x$, and voting equilibrium $\sigma$, if $x \in A_\sigma$, then $B(y, d(y, x) - 2r(y)) \subseteq A_\sigma$.

Lemma 2 is crucial. For starters, it allows us to address the second difficulty to generalizing the approach illustrated in the one-dimensional model: Instead of an exact distance $\tilde{d}$ of optimal proposals from the median, as in equation (2), Lemma 2 implies upper and lower bounds on the distance of optimal proposals from any point $y$. Specifically, if $B$ is the distance of the furthest socially-acceptable policy from $y$, then $\hat{x}$’s equilibrium proposal cannot be further than a distance $\tilde{d}(B, \hat{x}, y)$ from $y$, where

$$\tilde{d}(B, \hat{x}, y) := \begin{cases} \min \{B, d(y, \hat{x}) + \min_x \{d(x, \hat{x}) \mid x \in B(y, B - 2r(y))\}\} & \text{if } B \geq 2r(y), \\ B & \text{if } B < 2r(y). \end{cases}$$

Observe that $\max_B (\tilde{d}(B, \hat{x}, y) - d(y, \hat{x})) = r(y) + \max\{r(y) - d(y, \hat{x}), 0\}$, which is attainable at $B = d(y, \hat{x}) + r(y)$ when $d(y, \hat{x}) > r(y)$. This is a conservative bound and it hedges against the possibility that $\hat{x}$ may propose alternatives that are further from $y$ than $\hat{x}$ is.
a. If \( x \) is the farthest socially acceptable policy from \( y \) (\( d(y, x) = \bar{B}_\sigma(y) \)), then every alternative in the open ball \( \mathcal{B}(y, \bar{B}_\sigma(y) - 2r(y)) \) is also socially acceptable.

b. Proposals by \( \hat{x} \in \mathcal{B}(y, \bar{B}_\sigma(y) - 2r(y)) \) are at their ideal; proposals by \( \hat{x}' \in \mathcal{B}(y, \bar{B}_\sigma(y) - 2r(y)) \setminus \mathcal{B}(y, \bar{B}_\sigma(y) - 2r(y)) \) are at maximal distance \( 2d(y, \hat{x}') + 2r(y) - \bar{B}_\sigma(y) \) from \( y \); proposals by \( \hat{x}'' \notin \mathcal{B}(y, \bar{B}_\sigma(y)) \) are no farther than \( \bar{B}_\sigma(y) \) from \( y \). Red points indicate most distant possible proposals.

Since by assumption \( B \) is the maximal distance of acceptable proposals from \( y \), \( \hat{x} \)’s proposals are contained in \( \mathcal{B}(y, B) \); and \( \hat{x} \) cannot be optimizing by proposing agreements further than \( d(y, \hat{x}) + r(y) \) from \( y \) if \( B > \max\{d(y, \hat{x}), r(y)\} + r(y) \), because agreements within \( \mathcal{B}(y, B - 2r(y)) \supseteq \mathcal{B}(y, \max\{d(y, \hat{x}), r(y)\} - r(y)) \) are acceptable, by Lemma 2. Assume on the other hand that every policy within distance \( B \) from \( y \) is socially-acceptable. We can similarly reason that \( \hat{x} \)’s proposal is no closer to \( y \) than

\[
d(B, \hat{x}, y) := \min\{B, d(y, \hat{x})\}.
\]

In this case, \( \hat{x} \) can pass her ideal if \( B > d(y, \hat{x}) \) and otherwise would not propose anything closer than \( B \) to \( y \) since a proposal at a distance exactly \( d(y, \hat{x}) - B \) from her ideal is available.
To put all these results together, for all $y$ and any quasi-equilibrium $\sigma$ define

$$\bar{B}_\sigma(y) = \inf\{d \geq 0 \mid A_\sigma \subseteq B(y, d)\},$$
$$B_\sigma(y) = \sup\{d \geq 0 \mid B^o(y, d) \subseteq A_\sigma\}.$$

$\bar{B}_\sigma(y)$ is the largest distance of acceptable policies from $y$; and $B_\sigma(y)$ is the smallest (infimum) distance from $y$ of non-acceptable agreements. In the one-dimensional, simple majority rule case with a unique median, $\hat{x}_m$, and $\delta < 1$, $\bar{B}_\sigma(\hat{x}_m) = B_\sigma(\hat{x}_m)$ uniquely solve equation (3). In Lemma 3, I use these quantities to bound optimal proposals generalizing equation (2) to an upper and lower bound (4).

**Lemma 3.** For all $y$ and quasi-equilibrium $\sigma$, $p_\hat{x}(\{\hat{x}\}) = 1$ for all $\hat{x} \in B^o(y, B_\sigma(y))$ and

$$d(B_\sigma(y), \hat{x}, y) \leq d(y, z) \leq \bar{d}(\bar{B}_\sigma(y), \hat{x}, y), \text{ for all } \hat{x} \text{ and all } z \in \text{Support}(p_\hat{x}).$$

To specify analogues of equation (3) in the general case, define

$$\bar{F}(B, y) = (B - r_w(y))^2 - \left((1 - \delta)(d(y, q) + r_w(y))^2 + \delta \int (\bar{d}(B, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x})\right),$$
$$F(B, y) = (B + r_p(y))^2 - \left((1 - \delta)(d(y, q) - r_p(y))^2 + \delta \int (d(B, \hat{x}, y) - \min\{B, r_p(y)\})^2 \pi(d\hat{x})\right).$$

Using Lemmas 1-3, we now establish:

**Theorem 1.** Let $\sigma$ be a quasi-equilibrium. For all $y$, $\bar{B}_\sigma(y)$ and $B_\sigma(y)$ are finite and satisfy $\bar{F}(\bar{B}_\sigma(y), y) \leq 0$ and $F(B_\sigma(y), y) \geq 0$.

Inequalities $F(B_\sigma(y), y) \leq 0$ and $F(B_\sigma(y), y) \geq 0$ must hold for any quasi-equilibrium $\sigma$. Therefore, the maximum (minimum) candidate bound that satisfies them define a conservative inner (outer) bound on equilibrium socially-acceptable agreements. The advertised
outer and inner bounds on socially-acceptable agreements are now defined as:

\[
\bar{B}^*(y) := \sup\{B \geq 0 \mid \bar{F}(B, y) \leq 0\},
\]

\[
\hat{B}^*(y) := \inf\{B \geq 0 \mid F(B, y) \geq 0\}.
\]

With these definitions in place, we can now state the main results of this section:

**Theorem 2.** *For every quasi-equilibrium \(\sigma\) and every \(y\), \(B^\sigma(y, \bar{B}^*(y)) \subseteq A_\sigma \subseteq B(y, \bar{B}^*(y)).\)*

Theorem 2 bounds equilibrium socially-acceptable agreements. But, in general, it implies stricter bounds on equilibrium outcomes because equilibrium proposals are often confined in a proper subset of the set of acceptable agreements. To state these more stringent bounds on equilibrium outcomes let \(\lambda_\sigma\) be the lottery over outcomes induced by \(\sigma\). We have:

**Corollary 1.** *For every quasi-equilibrium \(\sigma\) and every \(y\),

\[
(Support(\pi) \cap B^\sigma(y, \hat{B}^*(y))) \subseteq Support(\lambda_\sigma) \subseteq B\left(y, \max_{\hat{x} \in Support(\pi)} d(\hat{B}^*(y), \hat{x}, y)\right).
\]

Note that \(\max_{\hat{x} \in Support(\pi)} d(\hat{B}^*(y), \hat{x}, y) \leq \hat{B}^*(y)\) and the inequality may be strict if \(\pi\) has bounded support. A second comment on Theorems 1 and 2 is that, unlike equation (3) which centers the motivating one-dimensional bound at the median, these Theorems establish generalizations centered at any, arbitrary, point \(y\). Corollary 2 highlights how this generality can be used to advantage to obtain even tighter bounds by combining bounds across different (all) \(y\). But if we are constrained to center the bounds at one point, Corollary 2 also proposes centering the inner bound at a \(P\)-center and the outer bound at a \(W\)-center.

**Corollary 2.** *For every quasi-equilibrium \(\sigma\) and every \(P\)-center \(y_p\) and \(W\)-center \(y_w\)

\[
B^\sigma(y_p, \hat{B}^*(y_p)) \subseteq \bigcup_y B^\sigma(y, \hat{B}^*(y)) \subseteq A_\sigma \subseteq \bigcap_y B(y, \bar{B}^*(y)) \subseteq B(y_w, \bar{B}^*(y_w)).
\]

Besides their theoretical significance, these results leave open the possibility that solutions to the inequations of Theorem 1 may not exist, or that the outer bound may
be infinite, or that either bound may be hard to compute. All are precluded by the next Theorem, which ensures that the bounds are well-defined and typically (in the space of model parameter values) obtainable as essentially unique solutions of the corresponding equations by elementary methods such as bisection.

**Theorem 3.** For all \( y \), \( \bar{B}^*(y), \bar{B}^*(y) \in \mathbb{R}_+ \) exist. Moreover,

1.a If \((1 - \delta)d(y, q) + r_w(y) > 0\), then either \( \bar{B}^*(y) = 2r(y) \) and \( \bar{F}(2r(y), y) > 0, \bar{F}(B, y) \neq 0 \) for all \( B > 0 \), or \( \bar{B}^*(y) \geq 2r_w(y) \) is the unique \( B > 0 \) that solves \( \bar{F}(B, y) = 0 \).

1.b If \((1 - \delta)d(y, q) + r_w(y) = 0\), then

i. If \( \delta = 1 \), \( \bar{F}(B, y) = 0 \) for all \( B \leq \bar{B}^*(y) = \max\left\{2r_p(y), \sup_B\{B \mid \pi(B(y, B - r_p(y))) = 0\}\right\} \).

ii. If \( \delta < 1 \), \( \bar{B}^*(y) = 0 \).

2.a If \((1 - \delta)(d(y, q) - r_p(y))^2 > r_p(y)^2\), then \( \bar{B}^*(y) > 0 \) uniquely solves \( \bar{F}(B, y) = 0 \).

2.b If \((1 - \delta)(d(y, q) - r_p(y))^2 \leq r_p(y)^2\), then \( \bar{B}^*(y) = 0 \).

Existence of the bounds follows from the fact that the inequalities are satisfied for at least one value and are diverging to positive infinity as the input variable increases. It may be tempting to discount the finding that the outer bound \( \bar{B}^*(y) \) is finite as obvious, but this result crucially relies on the structure of the set of socially acceptable agreements established in Lemma 2 and the finiteness of the second moments of the distribution of proposers \( \pi \). That the latter assumption cannot be disposed with is shown in Online Appendix B. One immediate implication of the finiteness of the outer bound \( \bar{B}^*(y) \) is that the set of equilibria of the game are identical to the set of equilibria of a game with compact policy space \( X' = \mathcal{B}(y, \bar{B}^*(y)) \), which in turn ensures that in the discounted case there do not exist profitable infinite deviations whenever there do not exist profitable one-stage deviations:

**Corollary 3.** If \( \delta < 1 \), every quasi-equilibrium is an equilibrium.

While the equations that yield the bounds are not typically continuously differentiable, they can be solved by bisection because they consistently change sign to the left and
right of a unique solution in the cases covered by parts 1.(a) and 2.(a) of the Theorem. In
the case of part 1.(a), $\bar{F}$ is shown to satisfy a single crossing condition in $[2r(y), +\infty)$; while
it is also possible that $\bar{F}(0, y) = 0$, the bound is strictly positive and uniquely obtainable in
$(0, +\infty)$ either as a unique solution to $\bar{F}(B, y) = 0$ or at $\bar{B}^*(y) = 2r(y)$. While the inner
bound is zero in all cases not covered by case 2.(a), the outer bound may be positive in
cases there is a continuum of non-zero solutions of equation $\bar{F}(B, y) = 0$, but these cases
are very precisely isolated by the conditions of part 1.(b)i of the Theorem. Therefore, the
major computation involved in obtaining these bounds is that of solving $(P_y)$ or $(W_y)$ or (if
the bounds are anchored at a $P$-center or $W$-center) solving $(P)$ or $(W)$. Either task can be
performed efficiently in the finite committee case, the more challenging (i.e., $(P)$ and $(W)$)
between the two by adapting the algorithm of Tovey (1992).

3 Implications & constitutional design

How tight are these bounds? What do they imply about the nature of the equilibrium
set and its relation to classic social choice concepts? How do they change with model
primitives? I take up these questions in this section. I have broken the discussion in small
segments.

Relation to the core & the uncovered set: By Theorem 3, if $y$ is the only acceptable
agreement in equilibrium ($\bar{B}^*(y) = 0$), it is necessary that $r_w(y) = 0$ and $y$ is a core point
(see Online Appendix D). The following Theorem elaborates:

Theorem 4. If $\bar{B}^*(y) = 0$ then $y$ is a core point. If $y$ is a core point, then $\bar{B}^*(y) = 0$ if

1. $\delta < 1$ and $q = y$, or,

\[
12\text{In the finite committee setting of size } N, (P_y) \text{ or } (W_y) \text{ are simpler problems and amount}
\text{to computing the distance from } \sum_{k=1}^{D} \binom{N}{k} \text{ possible binding hyperplanes.}
\]
2. \( \delta = 1, \ y \text{ is a unique core point, and } \pi(B(y, \epsilon)) > 0 \text{ for all } \epsilon > 0. \)

Therefore, we have a unique equilibrium with a policy at a core point if either, players are not perfectly patient but the status quo policy is at that core point (part 1); or, if the core is unique, players are perfectly patient, and there exist proposers to advocate for the core (part 2). This last condition on \( \pi \) is a parallel to the requirement that all players have positive recognition probability in the core equivalence and core selection analysis of Banks and Duggan (2000, 2006) for finite committees, which these results recover and complement.

Instead of requiring the core to be the only equilibrium outcome, we may instead consider whether outcomes near the core or near the uncovered set prevail when the core is empty. For all \( y \), the set of core points is contained in \( B(y, r_p(y)) \) while the uncovered set is contained in \( B(y, 2r(y)) \) (see Online Appendix D).\(^{13}\) Relying on Lemma 3, we establish a lower bound on the probability that equilibrium outcomes fall within the more conservative of these envelopes.

**Theorem 5.** For all \( y \) and every quasi-equilibrium \( \sigma, \lambda_\sigma (B(y, 2r(y))) \geq \pi (B(y, r(y))). \) If \( y \) is a core point, then \( \lambda_\sigma (B(y, 2r_p(y))) \geq \pi (B(y, r_p(y))). \)

Thus, if possible proposers are tightly concentrated within half the distance of the envelope on the uncovered set from its center, then equilibrium outcomes are also concentrated within the generalized uncovered set envelope. Combined with Theorem 4 and the forthcoming Theorem 8, these results generally imply that the prevalence of outcomes in or near these sets generally relies on the degree to which they are advocated by proposers.

**Status Quo:** A classic insight of political theory is that collective choices differ systematically with the location of the status quo. This is an almost immediate conclusion in static models of choice and Romer and Rosenthal (1978) is an elementary but forceful demonstration of that effect. The following result establishes the effect of the status quo on the equilibrium bounds.

\(^{13}\)The latter envelope that generalizes McKelvey’s simple majority rule bound on the uncovered set (McKelvey (1986)).
Theorem 6. For all \( y, \bar{B}^*(y) \) and \( B^*(y) \) weakly increase as \( d(y,q) \) increases and, if \( \delta < 1 \),

\[
\lim_{d(y,q)\to+\infty} B^*(y) = +\infty \quad \text{and} \quad \lim_{d(y,q)\to+\infty} (B^*(y) - B^*(y)) = (1 + \sqrt{1-\delta}) r(y).
\]

Both the set of possible equilibrium outcomes and the set of outcomes guaranteed to be socially acceptable in every equilibrium weakly expand as the status quo moves further away from the center \( y \). In the discounted case, these sets converge and cover the entire policy space as the distance of the status quo from \( y \) goes to infinity, but the difference between the outer and inner bounds converges to a constant at most twice the sum of the radii of the \( y \) centered \( P \)- and \( W \)-balls. Combining Theorem 6 and Lemma 3 we conclude that if the distribution of proposers has bounded support and the status quo is bad enough, then there exists a unique equilibrium in which all proposers propose their ideal.

Corollary 4. If \( \delta < 1 \) and \( \pi(B(y, \bar{P})) = 1 \) for some \( y, \bar{P} > 0 \), then there exists \( \bar{Q} > 0 \) such that for all \( q \not\in B(y, \bar{Q}) \) equilibrium is unique and satisfies \( p_\hat{x}(\{\hat{x}\}) = 1 \) for all \( \hat{x} \in \text{Support}(\pi) \).

Equilibrium Uniqueness: Besides Corollary 4, we also deduce a unique equilibrium whenever \( \bar{B}^*(y) = \bar{B}^*(y) \), whence \( A_\sigma = B(y, \bar{B}^*(y)) \) from the definition of the bounds. Theorem 4 provides one set of sufficient conditions for a unique equilibrium at the core. More generally:

Theorem 7. If \( B^*(y) = \bar{B}^*(y) \) then \( y \) is a core point and there exists a unique equilibrium, \( \sigma \), and \( \sigma \) is in pure strategies with \( A_\sigma = B(y, \bar{B}^*(y)) \). Furthermore, if \( y \) is a core point

1. \( B^*(y) = \bar{B}^*(y) = 0 \) in cases 1. and 2. of Theorem 4.

2. \( B^*(y) = \bar{B}^*(y) > 0 \) if and only if \( y \) is the unique core point and \( (1 - \delta)d(y, q) > 0 \).

Thus, in the discounted case, equilibrium is unique when the core is non-empty (not necessarily a singleton) and the status quo is a core point; the equilibrium in those cases puts mass one on the status quo (case 1 of Theorem 4). We also have a unique equilibrium with multiple socially acceptable agreements if the core is unique and the status quo is different
than the core (part 2 of Theorem 7). In the undiscounted case ($\delta = 1$) the conditions replicate those in part 2 of Theorem 4. Part 2 of Theorem 7 extends the uniqueness result of Cho and Duggan (2003) to more than one dimensions and infinite committees.

Proposal rights: While Theorems 2 and 3 assure us that equilibrium outcomes are bounded for any given distribution of proposers, $\pi$, the actual bounds generally vary as the allocation of proposal rights changes. Let $\Pi$ denote the space of all admissible distributions over proposers, that is, those with finite second moments. We can show:

Theorem 8. For all $y$:

1. $\bar{B}^*(y)$ weakly increases if $\pi$ changes to $\pi'$ such that $\pi' (B(y,d)) \leq \pi (B(y,d))$ for all $d$.

2. For all $\delta < 1$, there exists $\bar{B}^*(\delta, y) \in \mathbb{R}_+$ such that $\max_{\pi \in \Pi} \bar{B}^*(y) = \bar{B}^*(\delta, y)$.

3. $\bar{B}^*(\delta, y) = d(y, q)$ if $y$ is a core point, and $\lim_{\delta \to 1} \bar{B}^*(\delta, y) = +\infty$ otherwise.

4. If the core is empty, for all $\bar{B} > 0$ and all sufficiently large $\delta \in (\frac{1}{2}, 1)$ there exists $\pi_\delta$ and equilibrium $\sigma_\delta$ with $\lambda_{\sigma_\delta} (X \setminus B(y, \bar{B})) \geq \frac{2\delta - 1}{\delta}$.

Part 1 of Theorem 8 establishes that as proposal rights are shifted away from any point $y$, in the sense of first-order-stochastic dominance, then the outer bound on socially acceptable policies across equilibria centered at $y$ weakly expands. This might allow for policies to reach arbitrarily distant areas of the policy space, but part 2 of the Theorem bars this possibility in the discounted case: With stationary proposal rules,\textsuperscript{14} there is a maximal distance from $y$ which equilibrium outcomes cannot exceed, no matter how far proposal

\textsuperscript{14}With non-stationary rules, a McKelvey (1976)-like chain of proposers (not proposals) would likely not respect such an outer bound. Nevertheless, it is unclear whether an appropriate analogue of the restriction on the second moments of the distribution of proposers imposed on such a Markovian recognition protocol might yield an analogous result.
rights are shifted away from $y$. That distance is

$$B^*(\delta, y) := \frac{1 + \delta}{1 - \delta} r_w(y) + \sqrt{(d(y, q) + r_w(y))^2 + \delta \left( \frac{2r_w(y)}{1 - \delta} \right)^2},$$

and it increases with the radius of the $y$-centered $W$-ball as well as with the distance of the status quo from $y$. The maximal distance also depends on the discount factor, and parts 3 and 4 of Theorem 8 clarify this dependence: If $y$ is a core point, then the distance is constant and equal to the distance of the status quo from $y$. This is consistent with the finding in the special environment of Primo (2002). But if $y$ is not a core point, then this maximal outer distance is going to infinity as the discount factor tends to one. While this bound itself does not guarantee that there exist equilibria with outcomes at that distance, part 4 of the Theorem ensures that there exist sequences of equilibria with outcomes whose distance from $y$ goes to infinity as the discount factor goes to one, whenever the core is empty ($r_w > 0$).

These findings paint a mixed picture of the impact of proposal rights on legislative bargaining outcomes. On the positive side, part 2 of Theorem 8 imposes a cap on how much proposal rights can influence collective choice for any given discount factor less than one. On the other hand, unless the core is empty, this cap may be too permissive and, in fact, it becomes non-binding at the limit as legislators get patient in the strong sense implied by parts 3 and 4 of the Theorem. While Theorem 8 does not fully address what happens in the limit case of perfectly patient legislators, this is rectified in the following Theorem by focusing on the special environment when there exists a unique proposer:

**Theorem 9.** Assume $\delta = 1$ and $\pi(\{\hat{x}\}) = 1$ for some $\hat{x}$. Then $B^*(y) = \max\{d(y, \hat{x}), r_p(y)\} + r_w(y) + r(y)$ for all $y$. Furthermore, there exists an equilibrium with $\text{Support}(\lambda_\sigma) = \{\hat{x}\}$ if and only if for all $\hat{x}'$ such that $u(\hat{x}; \hat{x}') < u(q; \hat{x}')$, $\mu_\ell(\{\hat{x}'\}) \leq 1 - m_\ell$ for all $\ell$.

Theorem 9 establishes very weak necessary and sufficient conditions for an arbitrary unique proposer to propose and pass her ideal point in an equilibrium. The condition requires that if there is a legislator with absolute veto power, then she prefers the ideal point of the proposer over the status quo. Therefore, if there is no legislator with absolute veto, then
for any ideal point for the unique proposer there exists an equilibrium in which she passes
her ideal with probability one. This condition is weaker than the sufficient condition of
Duggan and Ma (2018) that there does not exist a restricted core point.\textsuperscript{15} The Theorem also
explicitly derives the outer bound on equilibrium acceptable policies, which is larger than
the distance of the unique proposer from $y$, for all possible proposer locations.

**Voting Rights:** What about the effect of voting rules and the assignment of voting rights?
These variables enter the equations that define equilibrium outcome bounds through the radii $r_p(y)$ and $r_w(y)$ of the $y$-centered $P$-ball and $W$-ball, and these quantities respond differently
to stricter (proper) voting rules: Radius $r_p(y)$ weakly increases (to, say, $r'_p(y)$) while $r_w(y)$
weakly decreases (to $r'_w(y)$) as the voting rule becomes more stringent by either requiring a
larger majority quota or the assent of an additional chamber. Theorem 10 establishes that
the introduction of a more stringent voting rule has ambiguous effect on the outer bound
though it weakly contracts both the inner bound and the maximal outer bound of equation
(5):

**Theorem 10.** Assume $m_{\ell^*} > \frac{1}{2}$ for some $\ell^*$. For all $y$, if

a. $m_{\ell}$ increases for any $\ell$, or

b. a chamber $\ell = L + 1$ is added with any voting weights $\mu_{L+1}$ and threshold $m_{L+1} \in (0, 1)$,

1. The inner bound $\bar{B}^*(y)$ weakly decreases.

2. The outer bound $\bar{B}^*(y)$ weakly decreases if $r(y) \geq r'(y)$, and may increase otherwise.

3. If $\delta < 1$, the maximal outer bound $\bar{B}^*(\delta, y)$ of Theorem 8 weakly decreases.

The comparative statics of Theorem 10 are mostly intuitive as they imply that more
stringent voting rules, requiring higher majority thresholds or the assent of more chambers,
\textsuperscript{15}Duggan and Ma (2018) look at limits of equilibria as $\delta \to 1$. Such limit equilibria
generally constitute a refined subset of the set of no-delay, stationary, subgame perfect
equilibria in stage-undominated voting strategies of that limit undiscounted game.
limit possible acceptable agreements across equilibria and proposer distributions in the discounted case (part 3); and also limit agreements that are guaranteed to be socially acceptable for any specific parameterization of the model (part 1). But, the outer bound and hence the set of possible equilibrium outcomes may decrease or increase (depending on how much $r_p(y)$ increases relative to the decrease of $r_w(y)$). Though paradoxical at first, this finding is reconcilable in view of the possibility that the set of agreements in any one equilibrium may indeed shrink with more stringent voting rules, but the set of equilibria may in fact expand leading to an expansion of the set of possible outcomes across equilibria. This is already anticipated by part 1(b)i of Theorem 3: When $\delta = 1$ and $r_w(y) = 0$, more stringent voting rules can lead to an increase from $r_p(y)$ to $r'_p(y)$ while $r_w(y) = r'_w(y) = 0$. If $\pi$ has full support in this case, then the new outer bound becomes $\bar{B}^* = 2r'_p(y)$ and the increase in the bound mirrors the increase in the radius of the $y$-centered $P$-ball.

4 Conclusion

I have established an inner and outer envelope on possible equilibrium acceptable agreements and outcomes in the spatial model of legislative bargaining. These bounds are easy to compute and provide a cheap alternative to computation of actual equilibria. They are informative and recover and extend several known results for uniqueness and core equivalence of equilibrium. They also allow comparative static analyses to address constitutional design questions. More stringent voting rules across legislative chambers exercise two countervailing forces on these bounds. On the one hand, they move the legislature closer to having a core or enlarge the set of core points, therefore exercise a centripetal force on equilibrium outcomes in any equilibrium. At the same time, the expansion of the set of core points may also result in a proliferation of equilibria so that, across equilibria, more policies may become possible. When it comes to the allocation of proposal rights, the analysis yields mixed conclusions. On the one hand, for any discount factor less than one, the equilibrium bounds dispel the possibility of arbitrarily distant alternatives prevailing in equilibrium, even if the distribution of possible proposers piles mass away from the center of the policy space. But
this outer limit on outcomes becomes more lax as legislators get patient and is not binding at the limit. These findings underscore the general point that the proper or ideal functioning of democratic institutions is not guaranteed with the mere assignment of equal or otherwise properly ‘weighted’ voting rights.

Several directions of improvement and extensions of these findings appear possible. Bounds analogous to those established in the present study could be developed with general concave preferences using the arguments of Cox (1987). Such bounds would likely be primarily of theoretical rather than practical value, unless strong conditions are imposed on the proximity of individual preferences to quadratic preferences. The analysis relies on generalizations of spherical envelopes on the set of alternatives socially preferred to a point as developed by McKelvey (1986), but he also established sharper versions of these envelopes that take the form of cardioids (McKelvey (1986), Lemma 8.1). That latter approach might be employed to advantage to produce sharper equilibrium bounds; similar improvement might be possible by tightening the conservative outer bound on optimal equilibrium proposals in (4). Lastly, with stationary recognition rules, the approach in this paper is particularly computationally simple, essentially relying on solving one equation in one unknown to derive the radius of the bounds. It is possible to obtain analogous bounds for non-stationary recognition rules, for example, allowing the identity of the proposer to depend on the realization of a state variable that follows a Markovian process, as in Kalandrakis (2004). With a finite number of states, it is a straightforward extension to derive a vector of equilibrium bounds for this model (one for each state) this time solving a square system of equations.

References


Appendix A  Proofs

As a background result, I state without proof a straightforward Lemma that makes use of mean-variance decomposition of expected quadratic payoffs, i.e., for all $\hat{x}$ and for

$$\mathbb{E}(\sigma) := (1 - \delta) q + \delta \int \int zp_\pi(dz)\pi(d\hat{x}),$$

$$\mathbb{V}(\sigma) := (1 - \delta) (q - \mathbb{E}(\sigma))^T \cdot (q - \mathbb{E}(\sigma)) + \delta \int \int (z - \mathbb{E}(\sigma))^T \cdot (z - \mathbb{E}(\sigma)) p_\pi(dz)\pi(d\hat{x}),$$

(6) 

$$v_\sigma(\hat{x}) = u(\mathbb{E}(\sigma); \hat{x}) - \mathbb{V}(\sigma).$$

**Lemma 4.** Consider any $x, \sigma$. If $x = \mathbb{E}(\sigma)$, $\{\hat{x} \mid u(x; \hat{x}) \geq v_\sigma(\hat{x})\} = \mathcal{I}$. If $x \neq \mathbb{E}(\sigma)$, there exists hyperplane $H$ perpendicular to the line segment $[x, \mathbb{E}(\sigma)]$, such that (a) $\{\hat{x} \mid u(x; \hat{x}) = v_\sigma(\hat{x})\} = H$, (b) $\{\hat{x} \mid u(x; \hat{x}) > v_\sigma(\hat{x})\} = H^+ \setminus H$, (c) $\{\hat{x} \mid u(x; \hat{x}) < v_\sigma(\hat{x})\} = H^- \setminus H$.

**Proof of Lemma 1.** Part 1. By Lemma 4, if $x \in A_\sigma$ there is (at least) a pivotal winning half-space that weakly prefer $x$. It intersects with $\mathcal{B}(y, r_w(y))$ by definition. Therefore, there exists $\hat{x} \in \mathcal{B}(y, r_w(y))$ such that $u(x; \hat{x}) \geq v_\sigma(\hat{x})$.

Part 2. Assume $u(x; \hat{x}) \geq v_\sigma(\hat{x})$ for all $\hat{x} \in \mathcal{B}(y, r_p(y))$. To get a contradiction, assume that $x \notin A_\sigma$. Then, by Lemma 4 $x \neq \mathbb{E}(\sigma)$ and there exists a hyperplane $H_{a,c}$ such that $H_{a,c} = \{\hat{x} \mid u(x; \hat{x}) = v_\sigma(\hat{x})\}$, and $H^+_{a,c} = \{\hat{x} \mid u(x; \hat{x}) \geq v_\sigma(\hat{x})\} \neq \mathcal{I}$. By $x \notin A_\sigma$, we must have $\mu_\ell(H^+_{a,c}) < m_\ell$ for some $\ell$. There exists a pivotal hyperplane $H_{a,c}(a)$ and, by definition, $H_{a,c}(a) \cap \mathbb{B}(y, r_p(y)) \neq \emptyset$. But $\mathcal{B}(y, r_p(y)) \subset H^+_{a,c}$, hence it must be that $c(a) > c$ and $H^+_{a,c}(a) \subset H^+_{a,c}$. This leads to a contradiction as there exists $\ell$ such that $m_\ell \leq \mu_\ell(H^+_{a,c}(a)) \leq \mu_\ell(H^+_{a,c}) < m_\ell$.

**Proof of Lemma 2.** Fix $y$, assume $x \in A_\sigma$, and consider any $x' \in \mathcal{B}(y, d(y, x) - 2r(y))$. We must show $x' \in A_\sigma$. By part 2 of Lemma 1 it suffices to show that $u(x'; \hat{x}') \geq v_\sigma(\hat{x}')$, for all $\hat{x}' \in \mathcal{B}(y, r_p(y))$. By Lemma 1 and $x \in A_\sigma$, there exists $\hat{x} \in \mathcal{B}(y, r_w(y))$ such that $u(x; \hat{x}) \geq v_\sigma(\hat{x})$. First we will show that $d(x', \hat{x}') \leq d(x, \hat{x}) - r(y)$ for all $\hat{x}' \in \mathcal{B}(y, r_p(y))$. 

1
Indeed, (recalling that \(r(y) = r_p(y) + r_w(y)\))

\[
d(x', \hat{x}') \leq d(y, \hat{x}') + d(x', y) \leq d(x', y) + r_p(y) \leq d(x, y) - r_p(y) - 2r_w(y) \leq d(x, \hat{x}) + d(\hat{x}, y) - r_p(y) - 2r_w(y) \leq d(x, \hat{x}) - r(y)
\]

[triangle inequality] \([\hat{x}' \in B(y, r_p(y))]\) \([x' \in B(y, d(x, y) - 2r(y))]\) \([triangle inequality]\) \([\hat{x} \in B(y, r_w(y))]\).

Now fix arbitrary \(\hat{x}' \in B(y, r_p(y))\); \(u(x'; \hat{x}') \geq v_\sigma(\hat{x}')\) follows because we have:

\[
u(x'; \hat{x}') = -d(x', \hat{x}')^2 \geq -d(x, \hat{x}) - r(y)^2 \geq -\left(\sqrt{-v_\sigma(\hat{x}) - r(y)}\right)^2 = -\left(\sqrt{d(\hat{x}, E(\sigma)) + d(\hat{x}, \hat{x}')^2 + V(\sigma) - r(y)}\right)^2 \geq -\left(\sqrt{(d(\hat{x}', E(\sigma)) + d(\hat{x}, y) + d(y, \hat{x}')^2 + V(\sigma) - r(y)}\right)^2 \geq -\left(\sqrt{(d(\hat{x}', E(\sigma)) + r(y)^2 + V(\sigma) - r(y)}\right)^2 = -\left(\sqrt{-v_\sigma(\hat{x}') + r(y)^2 + 2r(y)\sqrt{-v_\sigma(\hat{x}') - r(y)}\right)^2 \geq -\left(\sqrt{-v_\sigma(\hat{x}') + r(y)}\right)^2 = v_\sigma(\hat{x}')^2.
\]

\(u(x; \hat{x}) \geq v_\sigma(\hat{x})\) \([d(x', \hat{x}') \leq d(x, \hat{x}) - r(y)]\) \([u(x; \hat{x}) \geq v_\sigma(\hat{x})]\) \([(6)]\) \([\hat{x} \in B(y, r_w(y)), \hat{x}' \in B(y, r_p(y))]\) \([\hat{x} \in B(y, r_w(y)), \hat{x}' \in B(y, r_p(y))]\) \([\hat{x} \in B(y, r_w(y)), \hat{x}' \in B(y, r_p(y))]\) \([\sqrt{-v_\sigma(\hat{x}') \geq d(x', E(\sigma))]}\)

\(\square\)

**Proof of Lemma 3.** Fix a quasi-equilibrium \(\sigma\) and \(y\). Consider any \(\hat{x} \in B^\sigma(y, B_\sigma(y))\). By definition of \(B_\sigma(y)\) and \(A_\sigma\), closed, \(\hat{x} \in A_\sigma\) so that \(\{\hat{x}\} = \arg\max_{x \in A_\sigma} u(x; \hat{x})\), therefore \(p_\hat{x}(\{\hat{x}\}) = 1\) and \(d(B_\sigma(y), \hat{x}, y) = d(y, \hat{x}) = d(y, z)\) for all \(z \in \text{Support}(p_\hat{x})\). On the other hand, if \(d(y, \hat{x}) > B_\sigma(y)\) for some \(\hat{x}\), then \(\max_{x \in A_\sigma} u(x; \hat{x}) \geq \max_{x \in B(y, B_\sigma(y))} u(x; \hat{x}) = -(d(y, \hat{x}) - B_\sigma(y))^2\). Therefore, there cannot exist \(z \in \text{Support}(p_\hat{x})\) such that \(d(y, z) < \)
\[d(B_\sigma(y), \hat{x}, y) = B_\sigma(y).\]

Combining the two cases we conclude \(d(y, z) \geq d(B_\sigma(y), \hat{x}, y)\) for all \(\hat{x}\). To complete the proof of (4), note that the right-hand-side inequality holds trivially when \(\bar{B}_\sigma(y) < 2r(y)\) or \(\bar{B}_\sigma(y) \leq d(y, \hat{x}) + r(y)\), whence \(\bar{d}(\bar{B}_\sigma(y), \hat{x}, y) = \bar{B}_\sigma(y)\). Furthermore, if \(\bar{B}_\sigma(y) \geq d(y, \hat{x}) + 2r(y)\), then \(\hat{x} \in A_\sigma\) by Lemma 2, therefore \(\text{Support}(p_{\hat{x}}) = \{\hat{x}\}\) and \(\bar{d}(\bar{B}_\sigma(y), \hat{x}, y) = d(y, \hat{x})\). Thus, it remains to consider the case \(d(y, \hat{x}) + r(y) \leq \bar{B}_\sigma(y) \leq d(y, \hat{x}) + 2r(y)\) and \(\bar{B}_\sigma(y) \geq 2r(y)\). Then, by Lemma 2, \(\max_{x \in A_\sigma} u(x; \hat{x}) \geq \max_{x \in B(y, B(y)) - 2r(y)} u(x; \hat{x}) = -(d(y, \hat{x}) - (\bar{B}_\sigma(y) - 2r(y))^2\). Now if \(d(y, z) > \bar{d}(\bar{B}_\sigma(y), \hat{x}, y) = 2d(y, \hat{x}) + 2r(y) - \bar{B}_\sigma(y)\), we have \(u(z; \hat{x}) = -d(z, \hat{x})^2 \leq -(d(y, z) - d(y, \hat{x}))^2 < -(d(y, \hat{x}) - (\bar{B}_\sigma(y) - 2r(y))^2\), therefore it cannot be that \(z \in \text{Support}(p_{\hat{x}})\).

\[\square\]

**Proof of Theorem 1.** Fix a quasi-equilibrium \(\sigma\) and \(y\). We start with showing that both \(\bar{B}_\sigma(y), B_\sigma(y)\) are finite. If not, since \(2r(y) \geq \bar{B}_\sigma(y) - B_\sigma(y)\) by Lemma 2, then \(B_\sigma(y) = +\infty\) and, by Lemma 3, \(p_{\hat{x}}(\{\hat{x}\}) = 1\) for all \(\hat{x}\). Then

\[v_\sigma(\hat{x}) = (1 - \delta)u(q; \hat{x}) + \delta \int u(\hat{x}', \hat{x})\pi(d\hat{x}') > -\infty,\]

for all \(\hat{x}\) (because \(\pi\) has finite first and second moments). Furthermore, by Lemma 1, for all \(k = 1, 2, \ldots\) there exists \(x_k \in A_\sigma\) and \(\hat{x}_k \in B(y, r_w(y))\) such that \(d(y, x_k) > k\) and \(u(x_k; \hat{x}_k) = v_\sigma(\hat{x}_k) > -\infty\). Since \(d(x_k, \hat{x}_k) \geq k - r_w(y)\), we have \(\lim_{k \to +\infty} u(x_k; \hat{x}_k) = -\infty\), yet \(\min_{\hat{x} \in B(y, r_w(y))} v_\sigma(\hat{x}) = -\infty\), a contradiction. Therefore, both \(\bar{B}_\sigma(y), B_\sigma(y)\) are finite.

Next, consider \(F(B_\sigma(y), y) \leq 0\). It holds for all \(B_\sigma(y) \in [0, 2r_w(y)]\), because then

\[(1 - \delta)(d(y, q) + r_w(y))^2 + \delta \int (\bar{d}(\bar{B}_\sigma(y), \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \geq r_w(y)^2 \geq (\bar{B}_\sigma(y) - r_w(y))^2.\]

To show \(\bar{F}(\bar{B}_\sigma(y), y) \leq 0\) for \(\bar{B}_\sigma(y) \geq 2r_w(y)\), observe that there exists \(x' \in A_\sigma\) such that \(d(y, x') = \bar{B}_\sigma(y)\), since \(A_\sigma\) is closed and bounded (because \(\bar{B}_\sigma(y)\) is finite). By the first part
of Lemma 1 there also exists \( \hat{x}' \in \mathcal{B}(y, r_w(y)) \) such that \( u(x', \hat{x}') \geq v_\sigma(\hat{x}') \). We now have

\[
(\bar{B}_\sigma - r_w(y))^2 \leq (\bar{B}_\sigma(y) - d(y, \hat{x}'))^2 \\
= (d(y, x') - d(y, \hat{x}'))^2 \\
\leq d(x', \hat{x}')^2 = -u(x'; \hat{x}') \\
\leq -v_\sigma(\hat{x}') \\
\leq (1 - \delta)(d(\hat{x}', y) + d(y, q))^2 \\
+ \delta \int \int (d(\hat{x}', y) + d(y, z))^2 p_\hat{z}(dz) \pi(d\hat{x}) \quad \text{[triangle inequality]} \\
\leq (1 - \delta)(r_w(y) + d(y, q))^2 \\
+ \delta \int \int (r_w(y) + d(y, z))^2 p_\hat{z}(dz) \pi(d\hat{x}) \\
\leq (1 - \delta)(d(y, q) + r_w(y))^2 \\
+ \delta \int \int (\bar{B}_\sigma(y), \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \quad \text{[(4)].}
\]

To show \( F(B_\sigma(y), y) \geq 0 \), let \( \epsilon_k = \frac{1}{k} \). By the definition of \( B_\sigma(y) \), for all \( k = 1, 2, \ldots, \) there exists \( x_k \in \mathcal{B}(y, B_\sigma(y) + \epsilon_k) \setminus \mathcal{B}(y, B_\sigma(y)) \) such that \( x_k \notin A_\sigma \). Since \( x_k \notin A_\sigma \), there also exists corresponding \( \hat{x}_k \in \mathcal{B}(y, r_p(y)) \) such that \( u(x_k; \hat{x}_k) < v_\sigma(\hat{x}_k) \), by the contra-positive of the second part of Lemma 1. Now the sequence \((x_k, \hat{x}_k)\) belongs in \((\mathcal{B}(y, B_\sigma(y) + 1) \setminus \mathcal{B}^\circ(y, B_\sigma(y))) \times \mathcal{B}(y, r_p(y))\), a compact set, so – by going to a subsequence if necessary – there is a limit \((x_k, \hat{x}_k) \to (x^*, \hat{x}^*) \in (\mathcal{B}(y, B_\sigma(y)) \setminus \mathcal{B}^\circ(y, B_\sigma(y))) \times \mathcal{B}(y, r_p(y))\).

Since \( u, v_\sigma \) are continuous in \( x, \hat{x} \), we conclude

\[
(7) \quad u(x^*; \hat{x}^*) \leq v_\sigma(\hat{x}^*).
\]
We now have
\[
(B_\sigma(y) + d(y, \hat{x}^*))^2 = (d(y, x^*) + d(y, \hat{x}^*))^2
\]
\[
\geq d(x^*, \hat{x}^*)^2
\]
\[
\geq -v_\sigma(\hat{x}^*)
\]
\[
= (1 - \delta)d(\hat{x}^*, q)^2 + \delta \int \int d(\hat{x}, z)^2 p_{\hat{x}}(dz)\pi(d\hat{x})
\]
\[
\geq (1 - \delta) (d(y, q) - d(y, \hat{x}^*))^2 + \delta \int \int (d(y, z) - d(y, \hat{x}^*))^2 p_{\hat{x}}(dz)\pi(d\hat{x})
\]
\[
[d(a, b) \geq |d(a, c) - d(c, b)|].
\]

The left-hand-side of the resulting inequality
\[
(B_\sigma(y) + d(y, \hat{x}^*))^2 - (1 - \delta) (d(y, q) - d(y, \hat{x}^*))^2 - \delta \int \int (d(y, z) - d(y, \hat{x}^*))^2 p_{\hat{x}}(dz)\pi(d\hat{x}) \geq 0,
\]
is increasing in \(d(y, \hat{x}^*)\) and (because \(d(y, \hat{x}^*) \leq r_p(y)\)) we obtain
\[
(B_\sigma(y) + r_p(y))^2 - (1 - \delta) (d(y, q) - r_p(y))^2 - \delta \int \int (d(y, z) - r_p(y))^2 p_{\hat{x}}(dz)\pi(d\hat{x}) \geq 0.
\]

Now \(F(B_\sigma(y), y) \geq 0\) follows because
\[
\int \int (d(y, z) - r_p(y))^2 p_{\hat{x}}(dz)\pi(d\hat{x}) = \int_{B^p(y, B_\sigma(y))} (d(y, \hat{x}) - r_p(y))^2 \pi(d\hat{x})
\]
\[
+ \int_{\mathbb{I}\setminus B^p(y, B_\sigma(y))} (d(y, z) - r_p(y))^2 p_{\hat{x}}(dz)\pi(d\hat{x})
\]
\[
\geq \int_{B^p(y, B_\sigma(y))} (d(y, \hat{x}) - r_p(y))^2 \pi(d\hat{x})
\]
\[
+ \int_{\mathbb{I}\setminus B^p(y, B_\sigma(y))} \min_{z: d(y, z) \geq d(B_\sigma(y), \hat{x})} (d(y, z) - r_p(y))^2 \pi(d\hat{x}).
\]
\[
\geq \int (d(B_\sigma(y), \hat{x}, y) - \min\{B_\sigma(y), r_p(y)\})^2 \pi(d\hat{x}). \quad \square
\]

**Proof of Theorem 2.** Follows from Theorem 1 and the definition of \(B^*(y), B^*(y)\). \(\square\)
Proof of Corollary 1. The first inclusion follows from Lemma 3, while the second from the fact that \( \max_{B \in [0, B^*(y)]} \bar{d}(B, \hat{x}, y) \leq \max\{r(y), d(y, \hat{x})\} + r(y). \)

Proof of Theorem 3. By the Lebesgue dominated convergence Theorem and because \( \pi \) has finite first and second moments,\(^{16}\) we have

\[
\lim_{B \to +\infty} \int (\bar{d}(B, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) = \int (d(y, \hat{x}) + r_w(y))^2 \pi(d\hat{x}) < +\infty, \quad \text{and}
\]

\[
\lim_{B \to +\infty} \int (d(B, \hat{x}, y) - \min\{B, r_p(y)\})^2 \pi(d\hat{x}) = \int (d(y, \hat{x}, y) - r_p(y))^2 \pi(d\hat{x}) < +\infty.
\]

It follows that \( \lim_{B \to +\infty} \bar{F}(B, y) = \lim_{B \to +\infty} F(B, y) = +\infty. \) This ensures that \( B^*(y) \) (possibly 0) exists. By the definition of \( \bar{d}, \bar{F}(\cdot, y) \) is continuous in \([0, 2r(y)]\) and in \([2r(y), +\infty)\), and \( \lim_{B \to 2r(y)^+} \bar{F}(2r(y), y) \) and \( \lim_{B \to 2r(y)^-} \leq \bar{F}(2r(y), y) \), due to the fact that \( \lim_{B \to 2r(y)^+} \bar{d}(B, \hat{x}, y) = \bar{d}(2r(y), \hat{x}, y) \) \( \leq 2r(y) = \lim_{B \to 2r(y)^-} \bar{d}(B, \hat{x}, y) \) for all \( \hat{x} \). Existence of \( B^*(y) \in \mathbb{R} \) then follows from the above because \( (1 - \delta)d(y, q) + r_w(y) > 0 \) implies \( \bar{F}(0, y) < 0 \) in case 1.(a), and \( \bar{F}(0, y) = 0 \) in case 1.(b).

We now proceed to show parts 1.(a)-2.(b), starting with 1.(a). So assume \((1 - \delta)d(y, q) + r_w(y) > 0\), which implies \( \bar{F}(0, y) < 0 \), therefore \( B^*(y) > 0 \). If there does not exist \( B > 0 \) such that \( \bar{F}(B, y) = 0 \), then \( \lim_{B \to 2r(y)^-} \leq 0 < \bar{F}(2r(y), y) \) and \( B^*(y) = 2r(y) \), because \( \lim_{B \to +\infty} \bar{F}(B, y) = +\infty \) and \( \bar{F}(\cdot, y) \) is continuous in \([0, 2r(y)]\) and in \([2r(y), +\infty)\).

To complete the proof, then, we need to show that if there exists \( \bar{B} > 0 \), \( \bar{F}(\bar{B}, y) = 0 \), it is unique. Distinguish two cases:

Case 1. \( \bar{B} \in (0, 2r(y)) \). When \( B \in (0, 2r(y)) \), \( \bar{F}(B, y) \) is the quadratic polynomial:

\[
(B - r_w(y))^2 - (1 - \delta)(d(y, q) + r_w(y))^2 - \delta(B - r_w(y))^2.
\]

Because \( \bar{F}(0, y) < 0 \), \( \delta < 1 \), we can only have at most one root \( \bar{B} \in (0, 2r(y)) \) and \( 0 < \lim_{B \to 2r(y)^-} \bar{F}(B, y) \leq \bar{F}(2r(y), y) \). By Lemma 5 in Appendix C, there is no solution in

\(^{16}\)In particular, \((\bar{d}(B, \hat{x}, y) + r_w(y))^2 \leq (d(y, \hat{x}) + 2r(y) + r_w(y))^2 \) and \((d(B, \hat{x}, y) - \min\{B, r_p(y)\})^2 \leq (d(y_p(y), \hat{x}))^2 \) for all \( B \).
\[2r(y), +\infty\) when \(\bar{F}(2r(y), y) > 0\. \\

Case 2. \(\bar{B} \in [2r(y), +\infty)\). Then there is no other solution by case 1 and Lemma 5.

Moving to case 1.(b), observe that \(r(y) = r_p(y)\) and

\[
\bar{F}(B, y) = \begin{cases} 
B^2 - \delta B^2 & \text{if } B < 2r_p(y), \\
B^2 - \delta \left( \int_{B(y, B-r_p(y))} \bar{d}(B,\hat{x},y)^2 \pi(d\hat{x}) + \int_{I\setminus B(y, B-r_p(y))} B^2 \pi(d\hat{x}) \right) & \text{if } B \geq 2r_p(y).
\end{cases}
\]

When \(\delta = 1\), 1.(b)i follows since \(\bar{F}(B, y) = 0\) for all \(B < 2r_p(y)\), while \(B > 2r_p(y)\) satisfies \(F(B, y) = 0\) if and only if \(\pi(B(y, B - r_p(y))) = 0\). Indeed, \(\bar{d}(B,\hat{x},y) < B\) if and only if \(\hat{x} \in B(y, B - r_p(y))\), \(B > 2r_p(y)\), whilst \(F(B, y) > 0\) for all \(B > 2r_p(y)\) such that \(\pi(B(y, B - r_p(y))) > 0\). Similarly, 1.(b)ii, follows when \(\delta < 1\), as \(\bar{F}(B, y) \geq 0\) and \(B = 0\) is the unique solution to \(F(B, y) = 0\).

Now consider part 2. It is the case that \(F(0, y) < 0\) if and only if \((1 - \delta)(d(y, q) - r_p(y))^2 > r_p^2(y)\), hence both cases 2.(a) and 2.(b) follow if \(F\) is strictly increasing when \((1 - \delta)(d(y, q) - r_p(y))^2 > r_p^2(y)\). This is indeed the case, because \(\delta < 1\) and the integrand is either constant in \(B\), or (when differentiable w.r.t. \(B\)) has derivative smaller in absolute value than 2\((B + r_p(y))\).

Proof of Theorem 4. \(y\) is a core point iff \(r_w(y) = 0\) (Theorem 12, Online Appendix D). By Theorem 3, if \(\bar{B}^*(y) = 0\) then \(r_w(y) = 0\) so \(y\) is a core point. Part 1 follows from parts 1(b) ii of Theorem 3; part 2 from part 1(b)i of Theorem 3.

Proof of Theorem 5. We only need to show the first part because \(r(y) = r_p(y)\) when \(y\) is a core point \((r_w(y) = 0\) by Theorem 12, Online Appendix D). By the definition of \(\bar{d}\) we have

\[
\lambda_\sigma(B(y, 2r(y))) = 1 \quad \text{if} \quad \bar{B}_\sigma^*(y) \leq 2r(y),
\]

\[
\lambda_\sigma(B(y, 2r(y))) \geq \pi \left( B(y, \frac{\bar{B}_\sigma^*(y)}{2}) \right) \quad \text{if} \quad 2r(y) < \bar{B}_\sigma^*(y) \leq 4r(y),
\]

\[
\lambda_\sigma(B(y, 2r(y))) \geq \pi \left( B(y, 2r(y)) \right) \quad \text{if} \quad \bar{B}_\sigma^*(y) > 4r(y).
\]
**Proof of Theorem 6.** Write \( \bar{F}(B, y; q), F(B, y; q), \bar{B}^*(y; q), \) and \( B^*(y; q) \), to make the dependence on \( q \) explicit. The result is immediate from the fact that if \( d(y, q) < d(y, q') \), then \( \bar{F}(B, y; q) \geq \bar{F}(B, y; q') \) so that the largest solution to \( \bar{F}(B, y; q) \leq 0 \) weakly increases. For the lower bound, the results holds trivially if \( B^*(y; q) = 0 \). If case 2(a) of Theorem 3 applies, then \( F(B, y; q) \geq F(B, y; q') \), and the smallest solution to \( F(B, y; q) \geq 0 \) strictly increases. Thus, if \( \delta < 1 \) and \( d(y, q) \) is large enough we have \( \lim_{d(y, q) \rightarrow +\infty} B^*(y) = +\infty \) and \( \bar{F}(\bar{B}^*(y; q), y; q) = 0 \) and \( F(B^*(y; q), y; q) = 0 \). From these equations we conclude

\[
\bar{B}^*(y; q) - B^*(y; q) = r(y) + \frac{\sqrt{(1 - \delta)(d(y, q) + r_w(y))^2 + \delta \int (\bar{d}(\bar{B}^*(y; q), \hat{x}, y) + r_w(y))^2 \pi(d\hat{x})}}{\sqrt{1 - \delta}(d(y, q) - r_p(y))^2 + \delta \int (d(B^*(y; q), \hat{x}, y) - r_p(y))^2 \pi(d\hat{x})}.
\]

The integrals converge to (finite) constants \( \int (d(y, \hat{x}) + r_w(y))^2 \pi(d\hat{x}) \) and \( \int (d(y, \hat{x}) - r_p(y))^2 \pi(d\hat{x}) \) as \( \bar{B}^*(y; q) \) and \( B^*(y; q) \) go to infinity. The limit of the right-hand-side as \( d(y, q) \rightarrow +\infty \) is independent of the value of these integrals and equals \( r(y) + \sqrt{1 - \delta} r(y) \).

**Proof of Theorem 7.** That the equilibrium is unique and in pure strategies follows trivially from the convexity of the unique possible \( A_\sigma \). To show that it is necessary for \( y \) to be a core point, recall that \( y \) is a core point iff \( r_w(y) = 0 \) and \( y \) is a unique core if \( r_p(y) = 0 \) (Theorem 11. 12, Online Appendix D). Now assume \( B^*(y) = \bar{B}^*(y) = B^* \) but \( r_w(y) > 0 \) to get a contradiction. By Theorem 3, 1(a), it must be that \( B^* \geq 2r_w(y) > 0 \), therefore it must also be that Theorem 3, 2(a) applies, i.e., \( (1 - \delta)(d(y, q) - r_p(y))^2 > r^2_p(y) \) and \( F(B^*, y) = \bar{F}(B^*, y) = 0 \), unless \( B^* = 2r(y) \) in which case \( F(B^*, y) = 0 < \bar{F}(B^*, y) \) (by Theorem 3, 1(a)). These inequations then imply

\[
(B^* - r_w(y))^2 \geq (1 - \delta)(d(y, q) + r_w(y))^2 + \delta \int (\bar{d}(B^*, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}),
\]

\[
(B^* + r_p(y))^2 = (1 - \delta)(d(y, q) - r_p(y))^2 + \delta \int (d(B^*, \hat{x}, y) - \min\{B^*, r_p(y)\})^2 \pi(d\hat{x}).
\]
Expanding the squares on the right-hand-side of both inequations, reorganizing, and combining inequations (using $\bar{d}(B^*, \hat{x}, y) \geq d(B^*, \hat{x}, y)$ for all $\hat{x}$) we conclude

$$(B^* - r_w(y))^2 - r_w^2(y) \geq (B^* + r_p(y))^2 - (1 - \delta)r_p^2(y) - \delta \min\{B^*, r_p(y)\}^2$$

which in turn implies

$$-2B^*r_w(y) \geq 2B^*r_p(y) + \delta(r_p^2(y) - \min\{B^*, r_p(y)\}^2).$$

But the left-hand-side is negative and the right-hand-side positive which is impossible. Therefore $r_w(y) = 0$. By the same arguments, it is necessary that $y$ is a unique core in part 2, as we similarly cannot have $\bar{B}^*(y) = \tilde{B}^*(y) = B^* > 0$, $r_w(y) = 0$ but $r_p(y) > 0$. For sufficiency in part 2, assume $r_w(y) = r_p(y) = 0$ and $(1 - \delta)d(y, q) > 0$. Then $\hat{F}(B^*, y) = F(B^*, y)$ for all $B^*$ and $\hat{F}(0, y) = F(0, y) = -(1 - \delta)(d(y, q))^2 < 0$, therefore $B^*(y) = \bar{B}^*(y) > 0$. \hfill $\square$

**Proof of Theorem 8.** Part 1. Write $\hat{F}(B, y; \pi)$ to make dependence on $\pi$ explicit. We have $\hat{F}(B, y; \pi) \geq \bar{F}(B, y; \pi')$, because $\bar{d}$ weakly increases with $d(y, \hat{x})$ so that

$$\int (\bar{d}(B, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \leq \int (\bar{d}(B, \hat{x}, y) + r_w(y))^2 \pi'(d\hat{x}).$$

Part 2. If $\pi$ is such that $\pi(B(y, B - r(y))) = 0$, then $\bar{d}(B, \hat{x}, y) = B$ for all $\hat{x}$ in $\pi$’s support (whether $B > r(y)$ or not). Then $\hat{F}(B, y; \pi) = 0$ takes the form

$$(B - r_w(y))^2 - (1 - \delta)(d(y, q) + r_w(y))^2 - \delta \int (B + r_w(y))^2 \pi(d\hat{x}) = 0,$$

which has a unique solution $\tilde{B}^*(\delta, y)$ given in equation (5). By Theorem 3, for all $\pi$ such that $\pi(B(y, \bar{B}^*(\delta, y) - r(y))) = 0$, $\bar{B}^*(y) = \tilde{B}^*(\delta, y)$ as the unique $B > 0$ that solves $\hat{F}(B, y; \pi) = 0$. By part 1, there cannot be a larger $\bar{B}^*(y)$ for some $\pi'$, because for any $\pi'$ such that $\pi'(B(y, \bar{B}^*(\delta, y) - r(y))) > 0$, there exists $\pi''$ such that $\pi''(B(y, \bar{B}^*(\delta, y) - r(y))) = 0$ and $\pi''(B(y, d)) \leq \pi'(B(y, d))$ for all $d$.

Part 3. Straightforward from equation (5).
Part 4. Fix $\bar{B} > 0$ and let $y_w$ be a $W$-center. Since the core is empty, $r_w > 0$ (Theorem 12, Online Appendix D). Consider $\delta \in (\frac{1}{2}, 1)$. Let $\bar{B} > \max\{\bar{B} + d(y, y_w), d(y_w, q) + 2r(y_w)\}$. There exist at least two pivotal hyperplanes $H_m$ tangent to $B(y_w, r_w)$ (otherwise $r_w$ does not solve $(P)$). Let the tangency points be $\hat{x}_m, m = 1, \ldots, M, M \geq 2^{17}$ and for each $m$ let $\hat{x}_m^p$ be a point corresponding to $\hat{x}_m$ with the property that $y_w, \hat{x}_m, \hat{x}_m^p$ are collinear, $d(\hat{x}_m^p, y_w) = \bar{B}$, and $d(\hat{x}_m^p, \hat{x}_m) = \bar{B} - r_w$. Also, let $q'$ be the point with the property that $y_w, q, q'$ are collinear and $d(y_w, q') = d(y_w, q)$. Let $\lambda_\delta$ be a discrete probability distribution with support on $\hat{x}_1^p, \ldots, \hat{x}_M^p$ and expectation $y_w$. Let $\pi_\delta$ be the compound lottery putting probability $(\frac{1-\delta}{\delta})$ on $q'$ and $(\frac{2\delta-1}{\delta})$ on $\lambda_\delta$. Let $\sigma_\delta$ be such that each of proposers $q', \hat{x}_1^p, \ldots, \hat{x}_M^p$ propose their ideal point. Then, for any $\hat{x}$ we have

\[
v_\sigma(\hat{x}) = (1 - \delta)u(q; \hat{x}) + \delta \left(\frac{1-\delta}{\delta}u(q'; \hat{x}) + \frac{2\delta-1}{\delta} \sum_{m} \lambda_\delta(\hat{x}_m^p)u(\hat{x}_m^p; \hat{x})\right) = u(y_w; \hat{x}) - 2(1 - \delta)d(y_w, q)^2 - (2\delta - 1)\bar{B}^2,
\]

because $E(\sigma_\delta) = y_w$ and $\mathbb{V}(\sigma_\delta) = 2(1 - \delta)d(y_w, q)^2 + (2\delta - 1)\bar{B}^2$. By Lemma 4, if $\hat{x}_m$ weakly prefers $\hat{x}_m^p$, then so does everyone in the pivotal winning half-space $H_m^+$, therefore $\hat{x}_m^p \in A_{\sigma_\delta}$. Indeed, as long as $\delta > \frac{\bar{B} - r_w}{\bar{B}}$, then for all $m$:

\[u(\hat{x}_m^p, \hat{x}_m) = -(\bar{B} - r_w)^2 > r_w^2 - (2\delta - 1)\bar{B}^2 \geq v_\sigma(\hat{x}_m).\]

Now, because $\hat{x}_m^p \in A_{\sigma_\delta}$ and $d(y_w, \hat{x}_m^p) = \bar{B} > d(y_w, q) + 2r(y_w)$, we also have $q' \in A_{\sigma_\delta}$ by Lemma 2. Therefore, $\sigma_\delta$ is a quasi-equilibrium that is also an equilibrium by Corollary 3, part 4. Furthermore, for all $m$, $d(y, \hat{x}_m^p) \geq d(y_w, \hat{x}_m^p) - d(y, y_w) > \bar{B}$.

\[\text{Proof of Theorem 9.}\] First, $\bar{B}^*(y) = d(y, \hat{x}) + r(y) + r_w(y)$ uniquely solves $\bar{F}(B, y) = 0$ when $d(y, \hat{x}) \geq r_p(y)$, while if $d(y, \hat{x}) < r_p(y)$, then $\lim_{B \to 2r(y)} \bar{F}(B, y) = 0 \leq \bar{F}(2r(y), y) > 0$ and $\bar{B}^*(y) = 2r(y)$ by Theorem 3. Next, we show that there exists a quasi-equilibrium

\[\text{Because } y_w, r_w \text{ solve } (P) \text{ the convex hull of these tangency points contains } y_w \text{ (if there is a continuum of such points, we may select } M = D + 1 \text{ with that property).}\]
profile \( \sigma \) such that \( \hat{x} \) proposes \( \hat{x} \) with probability 1 and \( A(\hat{x}') = \{ x \mid u(x; \hat{x}') \geq u(\hat{x}; \hat{x}') \} \) for all \( \hat{x}' \). Trivially, \( \hat{x} \in A(\hat{x}') \) for all \( \hat{x}' \), therefore \( \hat{x} \in A_\sigma \) and \( \sigma \) satisfies \((E_v),(E_p)\). Since proposal \( \hat{x} \) is unanimously accepted, voter \( \hat{x}' \neq \hat{x} \) has a unilateral infinite deviation that possibly induces a different payoff only if \( \hat{x}' \) has absolute veto, i.e., \( \mu_\ell(\{\hat{x}''\}) > 1 - m_\ell \) for some \( \ell \). But such an infinite deviation by vetoer \( \hat{x}' \) is profitable if and only if \( v_\sigma(\hat{x}') = u(\hat{x}; \hat{x}') < u(q; \hat{x}') \).

Thus the stated conditions are indeed necessary and sufficient.

\[ \square \]

**Proof of Theorem 10.** By Theorem 14 in Online Appendix D, \( r_w(y) \) weakly decreases and \( r_p(y) \) weakly increases with the stated changes in the voting rule. Write \( \bar{F}(B, y; r_w(y), r(y)) \) and \( F(B, y; r_p(y)) \) to indicate the relevant parameters that enter each inequality. The results follow from the fact that

\[
\begin{align*}
\bar{F}(B, y; r_w(y), r(y)) &\geq \bar{F}(B, y; r'_w(y), r(y)) \quad \text{if } r_w(y) < r'_w(y), \\
\bar{F}(B, y; r_w(y), r(y)) &\geq \bar{F}(B, y; r_w(y), r'(y)) \quad \text{if } r(y) < r'(y), \text{ and,} \\
F(B, y; r_p(y)) &\leq F(B, y; r'_p(y)) \quad \text{if } r_p(y) < r'_p(y).
\end{align*}
\]

The first and last of these inequalities follow directly by taking derivatives (note that \( \bar{d} \) is unchanged when \( r(y) \) is and \( d \) does not depend on \( r_p(y) \)). The second follows because \( \bar{d}(B, \hat{x}, y) \) weakly increases with \( r(y) \). Parts 1 and 2 now follow from the above inequalities because \( \bar{B}^*(y) \) (and \( B^*(y) \)) is the largest (respectively, smallest) solution of \( \bar{F}(B, y) \leq 0 \) \((F(B, y) \geq 0)\). Part 3 follows directly from (5).

\[ \square \]

**Appendix B**  **Online – Equilibrium with unbounded support when \( \pi \) does not have finite second moments**

Let \( D = L = 1, \delta \in (0, 1) \). Assume \( \mu_1 \) has finite support with and odd number of \( K \geq 3 \) atoms \( \hat{x}_1, \ldots, \hat{x}_K \) and \( \mu_1(\{\hat{x}_k\}) = \frac{1}{K} \). Let \( \pi \) a Student’s \( t \) distribution with 2 degrees
of freedom. The strategy profile $\sigma$ with $p_\sigma(\{\hat{x}\}) = 1$ and $A(\hat{x}) = X = \mathbb{R}$ for all $\hat{x}$ is an equilibrium as $v_\sigma(\hat{x}) = -\infty$ for all $\hat{x}$. Conditions $(E_v)$ and $(E_p)$ are obviously met and there are no profitable infinite deviations as no voting atom $\hat{x}_k$ has veto over proposals. Therefore, $\bar{B}^*(y) = \bar{B}_\sigma(y) = +\infty$ for all $y$.

**Appendix C  Online – Single-crossing of upper bound equation**

**Lemma 5.** If $\bar{F}(\bar{B}, y) \geq 0$ for some $\bar{B} \geq 2r(y)$, then there is no $B' > \bar{B}$ such that $\bar{F}(B', y) = 0$.

*Proof.* Let $\bar{B} \geq 2r(y)$ be such that $\bar{F}(\bar{B}, y) \geq 0$, or

$$
(\bar{B} - r_w(y))^2 - (1 - \delta)(d(y, q) + r_w(y))^2 - \delta \int_{B(y, \bar{B} - r)} (\bar{d}(\bar{B}, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \\
- \delta \int_{I \setminus \mathcal{B}(y, \bar{B} - r(y))} (\bar{B} + r_w(y))^2 \pi(d\hat{x}) = \alpha \geq 0.
$$

Setting $A = -(1 - \delta)(d(y, q) + r_w(y))^2 - \delta \int_{\mathcal{B}(y, \bar{B} - r(y))} (\bar{d}(\bar{B}, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) - \alpha$, we obtain

$$
\delta \left(1 - \pi(\mathcal{B}(y, \bar{B} - r(y)))\right) = \frac{(\bar{B} - r_w(y))^2 - A}{(\bar{B} + r_w(y))^2} \leq \frac{(\bar{B} - r_w(y))^2}{(\bar{B} + r_w(y))^2}.
$$
Now suppose, to get a contradiction, that there exists $B' > \bar{B}$ such that $\bar{F}(B', y) = 0$. Because, $\bar{d}(\bar{B}, \hat{x}, y) \geq \bar{d}(B', \hat{x}, y)$ when $\hat{x} \in \mathcal{B}(y, \bar{B} - r(y))$,

\[
\int_{\mathcal{B}(y, B - r(y))} (\bar{d}(\bar{B}, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \geq \int_{\mathcal{B}(y, B - r(y))} (\bar{d}(B', \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}).
\]

As a consequence, also using $\bar{F}(\bar{B}, y) \geq \bar{F}(B', y) = 0$, we must have

\[
(\bar{B} - r_w(y))^2 - \delta \int_{I \setminus \mathcal{B}(y, B - r(y))} (\bar{d}(\bar{B}, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \geq (B' - r_w(y))^2 - \delta \int_{I \setminus \mathcal{B}(y, B - r(y))} (\bar{d}(B', \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}),
\]

In turn, because $B' \geq \bar{d}(B', \hat{x}, y)$ and $\bar{B} = \bar{d}(\bar{B}, \hat{x}, y)$ for all $\hat{x} \notin \mathcal{B}(y, \bar{B} - r(y))$, we have

\[
(\bar{B} - r_w(y))^2 - \delta \int_{I \setminus \mathcal{B}(y, B - r(y))} (\bar{d}(\bar{B}, \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}) \geq (B' - r_w(y))^2 - \delta \int_{I \setminus \mathcal{B}(y, B - r(y))} (\bar{d}(B', \hat{x}, y) + r_w(y))^2 \pi(d\hat{x}),
\]

or

\[\delta \left(1 - \pi(\mathcal{B}(y, \bar{B} - r(y)))\right) \left((B' + r_w(y))^2 - (\bar{B} + r_w(y))^2\right) \geq (B' - r_w(y))^2 - (\bar{B} - r_w(y))^2.\]

Combining inequalities, and using the fact that $B' > \bar{B} \geq 2r(y)$, we have

\[
\frac{(\bar{B} - r_w(y))^2}{(\bar{B} + r_w(y))^2} \geq \delta \left(1 - \pi(\mathcal{B}(y, \bar{B} - r(y)))\right) \geq \frac{(B' - r_w(y))^2 - (\bar{B} - r_w(y))^2}{(B' + r_w(y))^2 - (\bar{B} + r_w(y))^2} > 0.
\]

\[\text{Specifically,}
\]

\[
\bar{d}(\bar{B}, \hat{x}, y) = d(y, \hat{x}) = \bar{d}(B', \hat{x}, y) \quad \text{if } d(y, \hat{x}) \leq \bar{B} - 2r(y),
\]
\[
\bar{d}(\bar{B}, \hat{x}, y) = 2d(y, \hat{x}) - \bar{B} + 2r(y) > d(y, \hat{x}) = \bar{d}(B', \hat{x}, y) \quad \text{if } \bar{B} - 2r(y) < d(y, \hat{x}) \leq \min\{\bar{B} - 2r(y), B' - 2r(y)\},
\]
\[
\bar{d}(\bar{B}, \hat{x}, y) = 2d(y, \hat{x}) - \bar{B} + 2r(y) > 2d(y, \hat{x}) - B' + 2r(y) = \bar{d}(B', \hat{x}, y) \quad \text{if } B' - 2r(y) < d(y, \hat{x}) \leq \bar{B} - r(y).
\]
We now obtain a contradiction, because

\[
\frac{(B' + r_w(y))^2 - (\bar{B} + r_w(y))^2}{(B + r_w(y))^2} \geq \frac{(B' - r_w(y))^2 - (B - r_w(y))^2}{(B - r_w(y))^2} \iff
\]

\[
\frac{(B' + r_w(y))^2}{(B + r_w(y))^2} \geq \frac{(B' - r_w(y))^2}{(B - r_w(y))^2} \iff
\]

\[
(B' + r_w(y))(\bar{B} - r_w(y)) \geq (B' - r_w(y))(\bar{B} + r_w(y)) \iff
\]

\[
\bar{B}B' - B'r_w(y) + r_w(y)\bar{B} - r_w^2(y) \geq \bar{B}B' + B'r_w(y) - r_w(y)\bar{B} - r_w^2(y) \iff
\]

\[
\bar{B} \geq B'.
\]

This completes the proof. \qed

**Appendix D  Online – Generalized Medians, Core, and Uncovered set**

This appendix states two results relating the generalized medians with the core and the uncovered set and are invoked in the proofs of the main Theorems. These results are of independent interest and belong in a companion working paper but are stated here for completeness. First, we have:

**Theorem 12.** 1. \( x \) is a core point if and only if \( r_w(x) = 0 \).

2. There exists a core point if and only if the \( W \)-ball radius is zero.

3. If \( x \) is a core point, then \( x \in B(y, r_p(y)) \) for all \( y \).

**Proof.** Part 1. Necessity: Assume \( x \) is a core point but \( r_w(x) > 0 \) to get a contradiction. By \( r_w(x) > 0 \), there exists pivotal hyperplane \( H_{a,c^*}(a) \) such that \( x \notin H_{a,c^*}^+(a) \). Let \( x' \) be the point on \( H_{a,c^*}(a) \) closest to \( x \). Now all \( \hat{x} \in H_{a,c}^+ \setminus H_{a,c} \supset H_{a,c^*}^+(a) \), where \( c = a \cdot \frac{r + r'}{2} \), strictly prefer \( x' \) over \( x \). Therefore \( x'Px \) because \( H_{a,c^*}^+(a) \) is pivotal winning half-space. This is impossible because \( x \) is a core point. Therefore, \( r_w(x) = 0 \).
Sufficiency: Assume \( r_w(x) = 0 \) but suppose \( x \) is not a core point. Then there exists \( x', a \) and winning coalition \( H_{a,c}^+ \setminus H_{a,c} \), where \( c = a \cdot \frac{x+x'}{2} \), that strictly prefer \( x' \) over \( x \). But that means there exists pivotal winning half-space \( H_{a,c'_{(a)}}^+ \subseteq H_{a,c}^+ \), which is impossible since \( x \notin H_{a,c'_{(a)}}^+ \) yet \( r_w(x) = 0 \). Thus it must be that \( x \) is a core point.

Part 2. Follows from part 1.

Part 3. Suppose there is a core point \( x \), yet it is not contained in \( P \)-ball \( B(y,r) \). Let \( x' \) be the point closest to \( x \) in \( B(y,r) \). Let \( H_{a,c} \) be the hyperplane perpendicular to the line segment \([x, x']\), crossing through \( \frac{x+x'}{2} \) (i.e., \( c = a \cdot \frac{x+x'}{2} \)), and satisfying \( B(y,r) \subset H_{a,c}^+ \). All \( \hat{x} \) in pivotal winning half-space \( H_{a,c'_{(a)}}^+ \subset H_{a,c}^+ \) strictly prefer \( x' \) over \( x \), which is impossible since \( x \) is a core point. \( \Box \)

For the next result I introduce some notation and recall a definition. Given the voting rule defined in (1), define the strict and weak social preference relations \( P \) and \( R \), comparing alternatives \( x, x' \in X \) as

\[
xRx' \iff \mu_\ell(\{\hat{x} \mid u(x, \hat{x}) \geq u(x', \hat{x})\}) \geq m_\ell \text{ for all } \ell,
\]

\[
xPx' \iff \mu_\ell(\{\hat{x} \mid u(x, \hat{x}) > u(x', \hat{x})\}) \geq m_\ell \text{ for all } \ell.
\]

Building on these definitions, let \( P(x) := \{y \mid yPx\} \) and \( R(x) := \{y \mid yRx\} \) denote the set of alternatives strictly and weakly socially preferred to \( x \), respectively.

**Definition 2.** \( x \) is covered if there exists \( y \) such that \( yPx, P(y) \subseteq P(x) \), and \( R(y) \subseteq R(x) \).

**Theorem 13.** For all \( y \), if \( x \notin B(y, 2r(y)) \) then \( y \) covers \( x \).

**Proof.** Fix \( y \) and consider any \( x \notin B(y, 2r(y)) \). It can be shown that for all \( z \) and all \( w \), \( B^\circ(w, d(w, z) - 2r_p(w)) \subseteq P(z) \subseteq R(z) \subseteq B(w, d(w, z) + 2r_w(w)) \), and, using \( d(y, x) > 2r(y) \), we have

\[
P(y) \subseteq R(y) \subseteq B(y, 2r_w(y)) \subset B^\circ(y, d(y, x) - 2r_p(y)) \subseteq P(x) \subseteq R(x),
\]

and \( yPx \), therefore \( y \) covers \( x \). \( \Box \)
Theorem 14. Assume $m_{\ell^*} > \frac{1}{2}$ for some $\ell^*$. If

a. $m_{\ell}$ increases for any $\ell$, or

b. A chamber $\ell = L + 1$ is added with any $\mu_{L+1}$ and $m_{L+1} \in (0, 1)$, then for all $y$

1. $r_p$ and $r_p(y)$ weakly increase.

2. $r_w$ and $r_w(y)$ weakly decrease.