INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

By W.E. Diewert. May, 2013.

CHAPTER 13: The Economic Approach to the Producer Price Index

1. Introduction

Producer Price Indexes provide price indexes to deflate parts of the system of national accounts. As is well known[1], there are three distinct approaches to the measurement of Gross Domestic Product:

- the production approach;
- the expenditure or final demand approach; and
- the income approach.

The production approach[2] to the calculation of nominal GDP involves calculating the value of outputs produced by an industry and subtracting the value of intermediate inputs (or intermediate consumption to use the national accounting term) used in the industry. This difference in value is called the industry's *value added*. Summing these industry estimates of value added leads to an estimate of national GDP. Producer Price Indexes are used to separately deflate both industry outputs and industry intermediate inputs. A Producer Price Index (PPI) is also used to deflate an industry's nominal value added into value added at constant prices.

The economic approach to the PPI begins not at the industry level, but at the *establishment* level. An establishment is the PPI counterpart to a *household* in the theory of the Consumer Price Index. An *establishment* is an economic entity that undertakes *production* or *productive activity* at a specific geographic location in the country and has the capability of providing basic accounting information on the prices and quantities of the outputs it produces and the inputs it uses during an accounting period. In this chapter, attention will be restricted to establishments that undertake production under a *for profit* motivation.

Production is an activity that transforms or combines material inputs into other material outputs (e.g., agricultural, mining, manufacturing or construction activities) or transports materials from one location to another. Production also includes storage activities, which in effect transport materials in the same location from one time period to another. Finally, production also includes the creation of services of all types.[3]

There are two major problems with the above definition of an establishment. The first problem is that many production units at a specific geographic locations do not have the

[1] See Eurostat, IMF, OECD, UN and the World Bank (1993) or Bloem, Dippelsman and Maehle (2001; 17).
[2] Early contributors to this approach include Bowley (1922; 2), Rowe (1927; 173), Burns (1930; 247-250) and Copeland (1932; 3-5).
[3] See Hill (1999) for a taxonomy for services.

capability of providing basic accounting information on their inputs used and outputs produced. These production units may simply be a division or a single plant of a large firm and detailed accounting information on prices may only be available at the head office (or not be available at all). If this is the case, the definition of an establishment is modified to include production units at a number of specific geographic locations in the country instead of just one location. The important aspect of the definition of an establishment is that it be able to provide accounting information on prices and quantities.[4] A second problem is that while the establishment may be able to report accurate quantity information, its price information may be based on *transfer prices* that are set by a head office. These transfer prices are *imputed prices* and may not be very closely related to market prices.[5]

Thus the problems involved in obtaining the "correct" commodity prices for an establishment are generally more difficult than the corresponding problems associated with obtaining market prices for households. However, in this chapter, these problems will be ignored and it will be assumed that representative market prices are available for each output produced by an establishment and for each intermediate input used by the same establishment for at least two accounting periods.[6]

The economic approach to PPIs requires that establishment output prices *exclude* any indirect taxes that the various layers of government might levy on the outputs produced by the establishment. The reason for excluding these indirect taxes is that firms do not get to keep these tax revenues even though they may collect them for governments. Thus these taxes are not part of establishment revenue streams. On the other hand, the economic approach to PPIs requires that establishment intermediate input prices *include* any indirect taxes that governments might levy on these inputs used by the establishment. The reason for including these taxes is that they are actual costs that are paid for by the establishment.

For the first sections of this chapter, an *output price index*, an *intermediate input price index* and a *value added deflator* will be defined for *a single establishment* from the economic perspective. In subsequent sections, aggregation will take place over establishments in order to define national counterparts to these establishment price indexes.

[4] In this modified definition of an establishment, an establishment is generally a smaller collection of production units than a *firm* since a firm may be multinational. Thus another way of defining an establishment for our purposes is as follows: an establishment is the smallest aggregate of national production units that is able to provide accounting information on its inputs and outputs for the time period under consideration.

[5] For many highly specialized intermediate inputs in a multistage production process using proprietary technologies, market prices may simply not exist. Furthermore, there are several alternative concepts that could be used to define transfer prices; see Diewert (1985) and Eden (1998).

[6] As was argued in Chapter 1, the most reasonable concept for the market price for each commodity produced by an establishment during the accounting period under consideration is the value of production for that commodity divided by the quantity produced during that period; i.e., the price should be a *narrowly defined unit value* for that commodity.

Some notation is required. Consider the case of an establishment that produces N commodities during 2 periods, say periods 0 and 1. Denote the period t *output price vector* by $p_y^t \equiv [p_{y1}^t,...,p_{yN}^t]$ and the corresponding period t *output quantity vector* by $y^t \equiv [y_1^t,...,y_N^t]$ for t = 0,1. Assume that the establishment uses M commodities as intermediate inputs during periods 0 and 1. An *intermediate input* is an input which is produced by another establishment in the country or is an imported (nondurable) commodity.[7] The period t *intermediate input price vector* is denoted by $p_x^t \equiv [p_{x1}^t,...,p_{xM}^t]$ and the corresponding period t *intermediate input quantity vector* by $x^t \equiv [x_1^t,...,x_M^t]$ for t = 0,1. Finally, it is assumed that that the establishment utilizes the services of K *primary inputs* during periods 0 and 1. The period t *primary input vector* utilized by the establishment during period t is denoted by $z^t \equiv [z_1^t,...,z_K^t]$ for t = 0,1.

Note that it is assumed that the list of commodities produced by the establishment and the list of inputs used by the establishment *remains the same* over the two periods for which a price comparison is wanted. In real life, the list of commodities used and produced by an establishment does not remain constant over time. New commodities appear and old commodities disappear. The reasons for this churning of commodities include the following ones:

- Producers substitute new processes for older ones in response to changes in relative prices and some of these new processes use new inputs.
- Technical progress creates new processes or products and the new processes use inputs that were not used in previous periods.
- There are seasonal fluctuations in the demand (or supply) of commodities and this causes some commodities to be unavailable in certain periods of the year.

The introduction of new commodities will be discussed in a later chapter as will seasonal commodities. In the present chapter, these complications are ignored and it is assumed that the list of commodities remains the *same* over the two periods under consideration. It will also be assumed that all establishments are present in both periods under consideration; i.e., there are no new or disappearing establishments.[8]

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[7] However, capital inputs or durable inputs are excluded from the list of intermediate inputs. A durable input is an input whose contribution to production lasts more than one accounting period. This makes the definition of a durable input dependent on the length of the accounting period. However, by convention, an input is classified as being durable if it lasts longer than 2 or 3 years. Thus an intermediate input is a nondurable input which is also not a primary input. Durable inputs are classified as primary inputs even if they are produced by other establishments. Other primary inputs include labor, land and natural resource inputs.

[8] Rowe (1927; 174-175) was one of the first economists to appreciate the difficulties faced by statisticians when attempting to construct price or quantity indexes of production: "In the construction of an index of production there are three inherent difficulties which, inasmuch as they are almost insurmountable, impose on the accuracy of the index, limitations, which under certain circumstances may be somewhat serious. The first is that many of the products of industry are not capable of quantitative measurement. This difficulty appears in its most serious form in the case of the engineering industry. ... The second inherent difficulty is that the output of an industry, even when quantitatively measurable, may over a series of years change qualitatively as well as quantitatively. Thus during the last twenty years there has almost certainly been a tendency towards an improvement in the average quality of the yarn and cloth produced by the cotton industry .... The third inherent difficulty lies in the inclusion of new industries which develop importance as the years go on." These three difficulties are still with us today: think of the difficulties involved in measuring the outputs of the insurance and gambling industries; an increasing number of

When convenient, the above notation will be simplified to match up with our previous notation used in earlier chapters. Thus when studying the output price index, p_y^t and y^t will be replaced by p^t and q^t; when studying the input price index, p_x^t and x^t will be replaced by p^t and q^t and when studying the value added deflator, the composite vector of output and input prices, $[p_y^t,p_x^{t]}]$, will be replaced by p^t and the vector of net outputs, $[y^t,-x^t]$ by q^t. Thus the appropriate definition for p^t and q^t depends on the context.

To most practitioners in the field, our basic framework, which assumes that detailed price and quantity data are available for each of the possibly millions of establishments in the economy, will seem to be utterly unrealistic. However, two answers can be directed back at this very valid criticism:

- The spread of the computer and the ease of storing transaction data has made the assumption that the statistical agency has access to detailed price and quantity data less unrealistic. With the cooperation of businesses, it is now possible to calculate price and quantity indices of the type studied in chapters 3 and 4 above using very detailed data on prices and quantities.[9]
- Even if it is not realistic to expect to obtain detailed price and quantity data for every transaction made by every establishment in the economy on a monthly or quarterly basis, it is still necessary to accurately specify the *universe* of transactions in the economy. Once the target universe is known, sampling techniques can be applied in order to reduce data requirements.

2. An Overview of the Chapter

In this section, a brief overview of the contents of this chapter will be given. In sections 3-6, the economic theory of the *output price index* for an establishment is outlined. This theory is primarily due to Fisher and Shell (1972) and Archibald (1977). Various bounds to the output price index are developed along with some useful approximations to the theoretical output price index.

After reviewing some mathematics in section 7, in section 8, Diewert's (1976) theory of *superlative indexes* is outlined. A superlative index is one that can be evaluated using observable price and quantity data but under certain conditions, it can give exactly the same answer that the theoretical output price index studied in section 3 would give.

In later sections, the theory of the output price index outlined in section 3 is adapted to apply to intermediate input price indexes (section 10) and to value added deflators (section 11). All of these theoretical economic indexes are developed for a single establishment or production unit that can provide detailed price and quantity data.

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industries produce outputs that are one of a kind and hence price and quantity comparisons are necessarily difficult if not impossible and finally, the huge increases in research and development expenditures by firms and governments have led to ever increasing numbers of new products and industries.

[9] An early study that computed Fisher ideal indexes for a distribution firm in Western Canada for 7 quarters aggregating over 76,000 inventory items is Diewert and Smith (1994).

In section 12, aggregation over establishments is undertaken to obtain in *the national output price index*, which will typically be the country's flagship PPI.[10] In section 13, the *national intermediate input price index* is studied and in section 14, aggregation over establishments or industries takes place to obtain the *national value added deflator*.

In section 15, relationships between the output price index, the intermediate input price index and the value added deflator are studied in more detail while section 16 looks at the double deflation method for constructing a real value added deflator. In section 17, the problem of aggregating establishment value added deflators into a national value added deflator is studied.

In section 18, the national value added deflator is related to the GDP deflator for components of final demand. In particular, we look for conditions that will imply that the two deflators are exactly equal to each other.

Finally, section 19 concludes this chapter with some additional material on midyear indexes. Recall that this type of index was introduced in the consumer context in section 5 of chapter 7.

3. The Fisher Shell Output Price Index and Observable Bounds

In this subsection, we present an outline of the theory of the output price index for a single establishment that was developed by Fisher and Shell (1972) and Archibald (1977). This theory is the producer theory counterpart to the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). These economic approaches to price indexes rely on the assumption of (competitive) *optimizing behavior* on the part of economic agents (consumers or producers). Thus in the case of the output price index, given a vector of output prices p^t that the agent faces in a given time period t, it is assumed that the corresponding hypothetical quantity vector q^t is the solution to a revenue maximization problem that involves the producer's production function f or production possibilities set. (Hereafter the terms value of output and revenue are used interchangeably, inventory changes being ignored).

In contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors q^0 and q^1 are independent of the two price vectors p^0 and p^1. In the economic approach, the period 0 quantity vector q^0 is determined by the producer's period 0 production function and the period 0 vector of prices p^0 that the producer faces and the period 1 quantity vector q^1 is determined by the producer's period 1 production function f and the period 1 vector of prices p^1.

Before the output price index is defined for an establishment, it is first necessary to describe the establishment's technology in period t. In the economics literature, it is traditional to describe the technology of a firm or industry in terms of a production

[10] Since we aggregate only over establishments that have a for profit motivation (which can include publicly owned enterprises), a better term for the national aggregate might be *the business sector output price index*.

function, which tells us what the maximum amount of output that can be produced using a given vector of inputs. However, since most establishments produce more than one output, it is more convenient to describe the establishment's technology in period t by means of a *production possibilities set*, S^t. The set S^t describes what output vectors q can be produced in period t if the establishment has at its disposal the vector of inputs $v \equiv [x,z]$, where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus if [q,v] belongs to S^t,[11] then the nonnegative output vector q can be produced by the establishment in period t if it can utilize the nonnegative vector v of inputs.

Let $p \equiv (p_1,\ldots p_N)$ denote a vector of positive output prices that the establishment might face in period t and let $v \equiv [x,z]$ be a nonnegative vector of inputs that the establishment might have available for use during period t. Then the establishment's *revenue function* using period t technology is defined as the solution to the following revenue maximization problem:

(1) $R^t(p,v) \equiv \max_q \{\sum_{n=1}^N p_nq_n : (q,v)\in S^t\}$.

Thus $R^t(p,v)$ is the maximum value of output, $\sum_{n=1}^N p_nq_n$, that the establishment can produce, given that it faces the vector of output prices p and given that the vector of inputs v is available for use, using the period t technology.[12]

The period t revenue function R^t can be used to define the establishment's *period t technology output price index* P^t between any two periods, say period 0 and period 1, as follows:

(2) $P^t(p^0,p^1,v) = R^t(p^1,v)/R^t(p^0,v)$

where p^0 and p^1 are the vectors of output prices that the establishment faces in periods 0 and 1 respectively and v is a reference vector of intermediate and primary inputs.[13] If N = 1 so that there is only one output that the establishment produces, then it can be shown that the output price index collapses down to the single output price relative between periods 0 and 1, p_1^1 / p_1^0. In the general case, note that the output price index defined by (2) is a ratio of hypothetical revenues that the establishment could realize, given that it has the period t technology and the vector of inputs v to work with. The numerator in (2)

[11] We write this as $[q,v]\in S^t$ in what follows.

[12] The function R^t is known as the *GDP function* or the *national product function* in the international trade literature (see Kohli (1978)(1991) or Woodland (1982). It was introduced into the economics literature by Samuelson (1953). Alternative terms for this function include: (i) the *gross profit function*; see Gorman (1968); (ii) the *restricted profit function*; see Lau (1976) and McFadden (1978); and (iii) the *variable profit function*; see Diewert (1973) (1974a) (1993). The mathematical properties of the revenue function are laid out in these references and in Diewert (1974b).

[13] This concept of the output price index (or a closely related variant) was defined by Fisher and Shell (1972; 56-58), Samuelson and Swamy (1974; 588-592), Archibald (1977; 60-61), Diewert (1980; 460-461) (1983; 1055) and Balk (1998; 83-89). Readers who are familiar with the theory of the true cost of living index will note that the output price index defined by (2) is analogous to the *true cost of living index* which is a ratio of cost functions, say $C(u,p^1)/C(u,p^0)$ where u is a reference utility level: R replaces C and the reference utility level u is replaced by the vector of reference variables (t,v). For references to the theory of the true cost of living index, see Konüs (1924), Pollak (1983) or the earlier chapters.

is the maximum revenue that the establishment could attain if it faced the output prices of period 1, p^1, while the denominator in (2) is the maximum revenue that the establishment could attain if it faced the output prices of period 0, p^0. Note that all of the variables in the numerator and denominator functions *are exactly the same, except that the output price vectors differ*. This is a defining characteristic of an *economic price index*: all environmental variables are held constant with the exception of the prices in the domain of definition of the price index.

Note that there are a wide variety of price indexes of the form (2) depending on which reference technology t and reference input vector v that is chosen. Thus there is not a single economic price index of the type defined by (2): there is an entire *family* of indexes.

Usually, interest lies in two special cases of the general definition of the output price index (2): (i) $P^0(p^0,p^1,v^0)$ which uses the period 0 technology set and the input vector v^0 that was actually used in period 0 and (ii) $P^1(p^0,p^1,v^1)$ which uses the period 1 technology set and the input vector v^1 that was actually used in period 1. Let q^0 and q^1 be the observed output vectors for the establishment in periods 0 and 1 respectively. If there is revenue maximizing behavior on the part of the establishment in periods 0 and 1, then observed revenue in periods 0 and 1 should be equal to $R^0(p^0,v^0)$ and $R^1(p^1,v^1)$ respectively; i.e., the following equalities should hold:

$$(3)\quad R^0(p^0,v^0) = \sum_{n=1}^{N} p_n^0 q_n^0 \qquad \text{and} \qquad R^1(p^1,v^1) = \sum_{n=1}^{N} p_n^1 q_n^1.$$

Under these revenue maximizing assumptions, Fisher and Shell (1972; 57-58) and Archibald (1977; 66) have shown that the two theoretical indexes, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$ described in (i) and (ii) above, satisfy the following inequalities (4) and (5):

$$\begin{aligned}(4)\quad P^0(p^0,p^1,v^0) &\equiv R^0(p^1,v^0)/R^0(p^0,v^0) && \text{using definition (2)}\\ &= R^0(p^1,v^0)/ \textstyle\sum_{n=1}^{N} p_n^0 q_n^0 && \text{using (3)}\\ &\geq \textstyle\sum_{n=1}^{N} p_n^1 q_n^0/\sum_{n=1}^{N} p_n^0 q_n^0 && \text{since } q^0 \text{ is feasible for the maximization}\\ &\qquad \text{problem which defines } R^0(p^1,v^0) \text{ and so } R^0(p^1,v^0) \geq \textstyle\sum_{n=1}^{N} p_n^1 q_n^0\\ &\equiv P_L(p^0,p^1,q^0,q^1)\end{aligned}$$

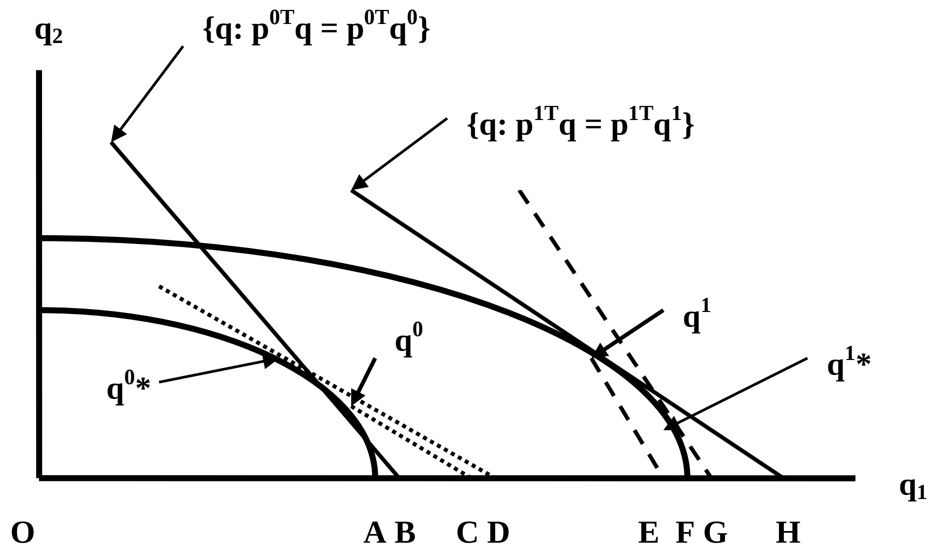
where P_L is the Laspeyres (1871) price index. Similarly,:

$$\begin{aligned}(5)\quad P^1(p^0,p^1,v^1) &\equiv R^1(p^1,v^1)/R^1(p^0,v^1) && \text{using definition (2)}\\ &= \textstyle\sum_{n=1}^{N} p_n^1 q_n^1 /R^1(p^0,v^1) && \text{using (3)}\\ &\leq \textstyle\sum_{n=1}^{N} p_n^1 q_n^1/\sum_{n=1}^{N} p_n^0 q_n^1 && \text{since } q^1 \text{ is feasible for the maximization}\\ &\qquad \text{problem which defines } R^1(p^0,v^1) \text{ and so } R^1(p^0,v^1) \geq \textstyle\sum_{n=1}^{N} p_n^0 q_n^1\\ &\equiv P_P(p^0,p^1,q^0,q^1)\end{aligned}$$

where P_P is the Paasche (1874) price index. Thus the inequality (4) says that the observable Laspeyres index of output prices P_L is a *lower bound* to the theoretical output price index $P^0(p^0,p^1,v^0)$ and the inequality (5) says that the observable Paasche index of output prices P_P is an *upper bound* to the theoretical output price index $P^1(p^0,p^1,v^1)$. Note that these inequalities are in the *opposite direction* compared to their counterparts in the theory of the true cost of living index.[14]

It is possible to illustrate the two inequalities (4) and (5) if there are only two commodities; see Figure 1 below, which is based on diagrams due to Hicks (1940; 120) and Fisher and Shell (1972; 57).

Figure 1: Bounds to the Paasche and Laspeyres Output Price Indexes



In Figure 1, the inequality (4) is illustrated for the case of two outputs that are both produced in both periods. The solution to the period 0 revenue maximization problem is the vector q^0 and the straight line through B represents the revenue line that is just tangent to the period 0 output production possibilities set, $S^0(v^0) \equiv \{(q_1,q_2,v^0)\in S^0\}$. The curved line through q^0 and A is the frontier to the producer's period 0 output production possibilities set $S^0(v^0)$. The solution to the period 1 revenue maximization problem is the vector q^1 and the straight line through H represents the revenue line that is just tangent to the period 1 output production possibilities set, $S^1(v^1) \equiv \{(q_1,q_2,v^1)\in S^1\}$. The curved line through q^1 and F is the frontier to the producer's period 1 output production possibilities set $S^1(v^1)$. The point q^{0*} solves the hypothetical maximization problem of maximizing revenue when facing the period 1 price vector $p^1 = (p_1^{\,1},p_2^{\,1})$ but using the period 0 technology and input vector. This is given by $R^0(p^1,v^0) = p_1^{\,1}q_1^{\,0*} + p_2^{\,1}q_2^{\,0*}$ and the dotted line through D is the corresponding isorevenue line $p_1^{\,1}q_1 + p_2^{\,1}q_2 = R^0(p^1,v^0)$. Note that the hypothetical revenue line through D is parallel to the actual period 1 revenue line through H. From (4), the hypothetical Fisher Shell output price index, $P^0(p^0,p^1,v^0)$, is

[14] This is due to the fact that the optimization problem in the cost of living theory is a cost *minimization* problem as opposed to our present revenue *maximization* problem. The method of proof used to derive (4) and (5) dates back to Konüs (1924), Hicks (1940) and Samuelson (1950).

$R^0(p^1,v^0)/[p_1^0q_1^0 + p_2^0q_2^0]$ while the ordinary Laspeyres output price index is $[p_1^1q_1^0 + p_2^1q_2^0]/[p_1^0q_1^0 + p_2^0q_2^0]$. Since the denominators for these two indexes are the same, the difference between the indexes is due to the differences in their numerators. In Figure 1, this difference in the numerators is expressed by the fact that the dotted revenue line through C lies *below* the parallel revenue line through D. Now if the producer's period 0 output production possibilities set were block shaped with vertex at q^0, then the producer would not change his or her production pattern in response to a change in the relative prices of the two commodities while using the period 0 technology and inputs. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dotted line through D would coincide with the dotted line through C and the true output price index $P^0(p^0,p^1,v^0)$, would *coincide* with the ordinary Laspeyres price index. However, block shaped production possibilities sets are not generally consistent with producer behavior; i.e., when the price of a commodity increases, producers generally supply more of it. Thus in the general case, there will be a gap between the points C and D. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *less* than the corresponding true output price index, $P^0(p^0,p^1,v^0)$.

Figure 1 can also be used to illustrate the inequality (5) for the two output case. Note that technical progress or increases in input availability have caused the period 1 output production possibilities set $S^1(v^1) \equiv \{(q_1,q_2) : [q_1,q_2,v^1] \text{ belongs to } S^1\}$ to be much bigger than the corresponding period 0 output production possibilities set $S^0(v^0) \equiv \{(q_1,q_2) : [q_1,q_2,v^0] \text{ belongs to } S^0\}$.[15] Secondly, note that the dashed lines through E and G are parallel to the period 0 isorevenue line through B. The point q^{1*} solves the hypothetical revenue maximization problem of maximizing revenue using the period 1 technology and inputs when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. This is given by $R^1(p^0,v^1) = p_1^0q_1^{1*} + p_2^0q_2^{1*}$ and the dashed line through G is the corresponding isorevenue line $p_1^1q_1 + p_2^1q_2 = R^1(p^0,v^1)$. From (5), the theoretical output price index using the period 1 technology and inputs is $[p_1^1q_1^1 + p_2^1q_2^1]/R^1(p^0,v^1)$ while the ordinary Paasche price index is $[p_1^1q_1^1 + p_2^1q_2^1]/[p_1^0q_1^1 + p_2^0q_2^1]$. Since the numerators for these two indexes are the same, the difference between the indexes is due to the differences in their denominators. In Figure 1, this difference in the denominators is expressed by the fact that the revenue line through E lies *below* the parallel cost line through G. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *greater* than the corresponding true output price index using current period technology and inputs, $P^1(p^0,p^1,v^1)$. Note that this inequality goes in the opposite direction to the previous inequality, (4). The reason for this change in direction is due to the fact that one set of differences between the two indexes takes place in the numerators of the two indexes (the Laspeyres inequalities) while the other set takes place in the denominators of the two indices (the Paasche inequalities).

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[15] However, validity of the inequality (5) does not depend on the relative position of the two output production possibilities sets. To obtain the strict inequality version of (5), we need two things: (i) we need the frontier of the period 1 output production possibilities set to be "curved" and (ii) we need relative output prices to change going from period 0 to 1 so that the two price lines through G and H in Figure 1 are tangent to *different* points on the frontier of the period 1 output production possibilities set.

There are two problems with the inequalities (4) and (5):

- There are *two* equally valid economic price indexes, $P^0(p^0,p^1,v^0)$ and $P^1(p^0, p^1 ,v^1)$, that could be used to describe the amount of price change that took place between periods 0 and 1 whereas the public will demand that the statistical agency produce a *single* estimate of price change between the two periods.
- Only *one sided* observable bounds to these two theoretical price indexes[16] result from this analysis and what are required for most practical purposes are *two sided* bounds.

In the following section, it will be shown how a possible solution to these two problems can be found.

4. The Fisher Ideal Index as an Approximation to an Economic Output Price Index

It is possible to define a theoretical output price index that falls *between* the observable Paasche and Laspeyres price indices. To do this, first define a hypothetical revenue function, $R(p,\alpha)$, that corresponds to the use of an α *weighted average* of the technology sets S^0 and S^1 for periods 0 and 1 as the reference technology and that uses an α weighted average of the period 0 and period 1 input vectors v^0 and v^1 as the reference input vector:

(6) $R(p,\alpha) \equiv \max_q \{\{\sum_{n=1}^N p_nq_n : [q,(1-\alpha)v^0 + \alpha v^1]\in[(1- \alpha)S^0 + \alpha S^1]\}$.

Thus the revenue maximization problem in (6) corresponds to the use of a weighted average of the period 0 and 1 input vectors v^0 and v^1 where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α and an "average" is used of the period 0 and period 1 technology sets where the period 0 set gets the weight $1-\alpha$ and the period 1 set gets the weight α , where α is a number between 0 and 1.[17] The meaning of the weighted average technology set in definition (6) can be explained in terms of Figure 1 as follows. As α changes continuously from 0 to 1, the output production possibilities set changes in a continuous manner from the set $S^0(v^0)$ (whose frontier is the curve which ends in the point A) to the set $S^1(v^1)$ (whose frontier is the curve which ends in the point F). Thus for any α between 0 and 1, a hypothetical establishment output production possibilities set is obtained which lies between the base period set $S^0(v^0)$ and the current period set $S^1(v^1)$. For each α, this hypothetical output production possibilities set can be used as the constraint set for a theoretical output price index.

The new revenue function defined by (6) is now used in order to define the following *family* (indexed by α) *of theoretical output price indexes*:

(7) $P(p^0,p^1,\alpha) \equiv R(p^1,\alpha)/R(p^0,\alpha)$.

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[16]The Laspeyres output price index is a lower bound to the theoretical index $P^0(p^0,p^1,v^0)$ while the Paasche output price index is an upper bound to the theoretical index $P^1(p^0,p^1,v^1)$.

[17] When α=0, $R(p,0) = R^0(p,v^0)$ and when $\alpha = 1$, $R(p,1) = R^1(p,v^1)$.

The important advantage that theoretical output price indexes of the form defined by (2) or (7) have over the traditional Laspeyres and Paasche output price indexes P_L and P_P is that these theoretical indexes deal adequately with *substitution effects*; i.e., when an output price increases, the producer supply should increase, holding inputs and the technology constant.[18]

Diewert (1983; 1060-1061) showed that, under certain conditions[19], there exists an α between 0 and 1 such that the theoretical output price index defined by (7) lies between the observable (in principle) Paasche and Laspeyres output indexes, P_P and P_L ; i.e., there exists an α such that

$$(8) \qquad P_L \le P(p^0,p^1,\alpha) \le P_P \quad \text{or} \quad P_P \le P(p^0,p^1,\alpha) \le P_L \,.$$

The fact that the Paasche and Laspeyres output price indexes provide upper and lower bounds to a "true" output price $P(p^0,p^1,\alpha)$ in (8) is a more useful and important result than the one sided bounds on the "true" indices that were derived in (4) and (5) above. If the observable (in principle) Paasche and Laspeyres indexes are not too far apart, then taking a symmetric average of these indexes should provide a good approximation to an economic output price index where the reference technology is somewhere between the base and current period technologies. Using an axiomatic approach, Proposition 2 in chapter 1 suggested that the geometric average was "best" and this result led to the geometric mean, the Fisher price index, P_F:

$$(9) \quad P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\, P_P(p^0,p^1,q^0,q^1)]^{1/2}.$$

Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical output price index.[20]

---

[18] This is a normal output substitution effect. However, empirically, it will often happen that observed period to period decreases in price are not accompanied by corresponding decreases in supply. However, these abnormal "substitution" effects can be rationalized as the effects of technological progress. For example, suppose the price of computer chips decreases substantially going from period 0 to 1. If the technology were constant over these two periods, we would expect domestic producers to decrease their supply of chips going from period 0 to 1. In actual fact, the opposite happens but what has happened is that technological progress has led to a sharp reduction in the cost of producing chips which is passed on to demanders of chips. Thus the effects of technological progress cannot be ignored in the theory of the output price index. The counterpart to technological change in the theory of the cost of living index is taste change, which is often ignored!

[19] The proof is essentially the same as the proof of Proposition 1 in chapter 4. Diewert adapted a method of proof due originally to Konüs (1924) in the consumer context. Sufficient conditions on the period 0 and 1 technology sets for the result to hold are given in Diewert (1983; 1105). Our exposition of the material in this chapter also draws on Chapter 2 in Alterman, Diewert and Feenstra (1999).

[20] It should be noted that Fisher (1922) constructed Laspeyres, Paasche and Fisher output price indexes for his U.S. data set. Fisher also adopted the view that the product of the price and quantity index should equal the value ratio between the two periods under consideration, an idea that he already formulated in Fisher (1911; 403). He did not consider explicitly the problem of deflating value added but by 1930, his ideas on deflation and the measurement of quantity growth being essentially the same problem had spread to the problem of deflating nominal value added; see Burns (1930).

The bounds given by (4), (5) and (8) are the best bounds that can be obtained on economic output price indexes without making further assumptions. In the next subsection, further assumptions are made on the two technology sets S^0 and S^1 or equivalently, on the two revenue functions, $R^0(p,v)$ and $R^1(p,v)$. With these extra assumptions, it is possible to determine the geometric average of the two theoretical output price indices that are of primary interest, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$.

5. The Törnqvist Index as an Approximation to an Economic Output Price Index

An alternative to the Laspeyres and Paasche indexes defined above in (4) and (5) or the Fisher index defined by (9) above is to use the Törnqvist (1936)(1937) Theil (1967) price index P_T, whose natural logarithm is defined as follows:

$$(10)\quad \ln P_T(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) \ln (p_n^1/p_n^0)$$

where $s_n^t = p_n^t q_n^t / \sum_{k=1}^{N} p_k^t q_k^t$ is the revenue share of commodity n in the total value of sales in period t.

Recall the definition of the period t revenue function, $R^t(p,v)$, defined earlier by (1) above. Now assume that the period t revenue function has the following *translog functional form*[21] ;i.e., for t = 0,1, assume that:

$$(11)\quad \ln R^t(p,v) = \alpha_0^t + \sum_{n=1}^{N} \alpha_n^t \ln p_n + \sum_{m=1}^{M+K} \beta_m^t \ln v_m + (1/2) \sum_{n=1}^{N} \sum_{j=1}^{N} \alpha_{nj}^t \ln p_n \ln p_j + \sum_{n=1}^{N} \sum_{m=1}^{M+K} \beta_{nm}^t \ln p_n \ln v_m + (1/2) \sum_{m=1}^{M+K} \sum_{k=1}^{M+K} \gamma_{mk}^t \ln v_m \ln v_k$$

where the α_n^t coefficients satisfy the restrictions:

$$(12)\quad \sum_{n=1}^{N} \alpha_n^t = 1 \quad \text{for } t = 0,1$$

and the α_{nj}^t and the β_{nm}^t coefficients satisfy the following restrictions:[22]

$$(13)\quad \sum_{n=1}^{N} \alpha_{nj}^t = 0 \quad \text{for } t = 0,1 \text{ and } j = 1,2,\ldots,N;$$
$$\sum_{n=1}^{N} \beta_{nm}^t = 0 \quad \text{for } t = 0,1 \text{ and } m = 1,2,\ldots,M.$$

The restrictions (12) and (13) are necessary to ensure that $R^t(p,v)$ is linearly homogeneous in the components of the output price vector p (which is a property that a revenue function must satisfy[23]). Note that at this stage of our argument the coefficients that characterize the technology in each period (the α's, β's and γ's) are allowed to be completely different in each period. It should also be noted that the translog functional form is an example of a *flexible* functional form[24]; i.e., it can approximate an arbitrary technology to the second order.

[21] This functional form was introduced and named by Christensen, Jorgenson and Lau (1971). It was adapted to the revenue function or profit function context by Diewert (1974a).
[22] It is also assumed that the symmetry conditions $\alpha_{nj}^t = \alpha_{jn}^t$ for all n,j and for t = 0,1 and $\gamma_{mk}^t = \gamma_{km}^t$ for all m,k and for t = 0,1 are satisfied.
[23] See Diewert (1973) (1974a) for the regularity conditions that a revenue or profit function must satisfy.
[24] The concept of flexible functional form was introduced by Diewert (1974a; 113).

A result in Caves, Christensen and Diewert (1982; 1410) can now be adapted to the present context: if the quadratic price coefficients in (11) are equal across the two periods of the index number comparison (i.e., $\alpha_{nj}^0 = \alpha_{nj}^1$ for all n,j), then the geometric mean of the economic output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the economic output price index that uses period 1 technology and the period 1 input vector v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by (10) above; i.e.,

(14) $P_T(p^0,p^1,q^0,q^1) = [P^0(p^0,p^1,v^0)\ P^1(p^0,p^1,v^1)]^{1/2}$.

The assumptions required for this result seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and our assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic output price indices and it corresponds to a flexible functional form, the Törnqvist output price index number formula is said to be *superlative,* following the terminology used by Diewert (1976).

In the following sections, additional superlative output price formulae are derived. However, this section concludes with a few words of caution on the applicability of the economic approach to Producer Price Indexes.

The above economic approaches to the theory of output price indexes have been based on the assumption that producers take the prices of their outputs as given fixed parameters that they cannot affect by their actions. However, a *monopolistic supplier* of a commodity will be well aware that the average price that can be obtained in the market for their commodity will depend on the number of units supplied during the period. Thus under noncompetitive conditions when outputs are monopolistically supplied (or when intermediate inputs are monopsonistically demanded), the economic approach to producer price indexes breaks down. The problem of modeling noncompetitive behavior does not arise in the economic approach to consumer price indexes because, usually, a single household does not have much control over the prices it faces in the marketplace.

The economic approach to producer output price indexes can be modified to deal with certain monopolistic situations. The basic idea is due to Frisch (1936; 14-15) and it involves linearizing the demand functions a producer faces in each period around the observed equilibrium points in each period and then calculating shadow prices which replace market prices. Alternatively, one can assume that the producer is a markup monopolist and simply adds a markup or premium to the relevant marginal cost of production.[25] However, in order to implement these techniques, econometric methods will usually have to be employed and hence, these methods are not really suitable for use by statistical agencies, except in very special circumstances when the problem of

[25] See Diewert (1993; 584-590) for a more detailed description of these techniques for modeling monopolistic behavior and for additional references to the literature.

noncompetitive behavior is thought to be very significant and the agency has access to econometric resources.

Problem

1. Let $R^t(p,v)$ be defined by (11) for t = 0,1 and define the logarithms of the two Fisher Shell output price indexes as

(i) $\ln P^0(p^0,p^1,v^0) \equiv \ln\{R^0(p^1,v^0)/R^0(p^0,v^0)\}$;
(ii) $\ln P^1(p^0,p^1,v^1) \equiv \ln\{R^1(p^1,v^1)/R^1(p^0,v^1)\}$.

Suppose that the following restrictions on the parameters of the two translog revenue functions hold:

(iii) $\alpha_{nj}{}^0 = \alpha_{nj}{}^1$ for all $1 \leq n,j \leq N$.

Show that

(iv) $\ln P^0(p^0,p^1,v^0) + \ln P^1(p^0,p^1,v^1) = [\nabla_{\ln p} \ln R^0(p^0,v^0) + \nabla_{\ln p} \ln R^1(p^1,v^1)]^T[\ln p^1 - \ln p^0]$.

Hint: You may find it useful to use Diewert's (1976; 118) quadratic identity studied in chapter 1, which implies the following two equations in the present context.:

(v) $\ln \{R^0(p^1,v^0)/R^0(p^0,v^0)\} = (1/2)[\nabla_{\ln p} \ln R^0(p^0,v^0) + \nabla_{\ln p} \ln R^0(p^1,v^0)]^T[\ln p^1 - \ln p^0]$;
(vi) $\ln \{R^1(p^1,v^1)/R^1(p^0,v^1)\} = (1/2)[\nabla_{\ln p} \ln R^1(p^0,v^1) + \nabla_{\ln p} \ln R^1(p^1,v^1)]^T[\ln p^1 - \ln p^0]$.

6. Homogeneous Separability and the Output Price Index

Instead of representing the period t technology by a set S^t, the period t technology is now represented by a *factor requirements function* F^t; i.e., $v_1 = F^t(q_1, q_2, \ldots ,q_N; v_2, v_3, \ldots ,v_{M+K})$ is set equal to the minimum amount of input 1 that is required in period t in order to produce the vector of outputs $q \equiv [q_1,...,q_N]$ given that the amounts $v_2, v_3, \ldots ,v_{M+K}$ of the remaining inputs are available for use. It is assumed that a linearly homogeneous aggregator function f exists for outputs; i.e., assume that functions f and G^t exist such that[26]

(15) $F^t(q,v_2,...,v_{M+K}) = G^t(f(q),v_2,...,v_{M+K})$; $\quad$ t = 0,1.

[26] This method for justifying aggregation over commodities is due to Shephard (1953; 61-71). It is assumed that f(q) is an increasing ,positive and convex function of q for positive q. Samuelson and Swamy (1974) and Diewert (1980; 438-442) also develop this approach to index number theory.

In technical terms, t outputs are said to be homogeneously weakly separable from the other commodities in the production function.[27] The intuitive meaning of the separability assumption that is defined by (15) is that an *output aggregate* $Q \equiv f(q_1,...,q_N)$ *exists*; i.e., a measure of the aggregate contribution to production of the amounts q_1 of the first output, q_2 of the second output, ..., and q_N of the Nth output is the number $Q = f(q_1,q_2,...,q_N)$. Note that it is assumed that the linearly homogeneous output aggregator function f does not depend on t. These assumptions are quite restrictive from the viewpoint of empirical economics[28] but strong assumptions are required in order to obtain the existence of output aggregates.[29]

It turns out that the *output aggregator function* f has a corresponding *unit revenue function*, r, defined as follows:

(16) $r(p) \equiv \max_q \{\sum_{n=1}^N p_n q_n : f(q) = 1\}$

where $p \equiv [p_1,...,p_N]$ and $q \equiv [q_1,...,q_N]$. Thus r(p) is the maximum revenue that the establishment can make, given that it faces the vector of output prices p and is asked to produce a combination of outputs $[q_1,...,q_N] = q$ that will produce a unit level of aggregate output.[30]

Let $Q > 0$ be an aggregate level of output. Then it is straightforward to show that[31]:

$$
\begin{aligned}
(17)\quad \max_q \{\textstyle\sum_{n=1}^N p_n q_n : f(q) = Q\} &= \max_q \{\textstyle\sum_{n=1}^N p_n q_n : (1/Q)f(q) = 1\} \\
&= \max_q \{\textstyle\sum_{n=1}^N p_n q_n : f(q/Q) = 1\} \\
&\qquad \text{using the linear homogeneity of f} \\
&= Q \max_q \{\textstyle\sum_{n=1}^N p_n q_n/Q : f(q/Q) = 1\} \\
&= Q \max_u \{\textstyle\sum_{n=1}^N p_n u_n : f(u) = 1\} \qquad \text{letting } u \equiv q/Q \\
&= Q\, r(p) \qquad \text{using definition (16).}
\end{aligned}
$$

Thus r(p)Q is the maximum revenue that the establishment can make, given that it faces the vector of output prices p and is asked to produce a combination of outputs $[q_1,...,q_N] = q$ that will produce the level Q of aggregate output.

---

[27] This terminology follows that used by Geary and Morishima (1973). The concept of weak separability dates back to Sono (1945) and Leontief (1947). A survey of separability concepts can be found in Blackorby, Primont and Russell (1978).

[28] Suppose that in period 0, the vector of inputs v^0 produces the vector of outputs q^0. Our separability assumptions imply that the same vector of inputs v^0 could produce *any* vector of outputs q such that $f(q) = f(q^0)$. In real life, as q varied, we would expect that the corresponding input requirements would also vary instead of remaining fixed.

[29] The assumptions on the technology of the establishment that we make in this section are considerably stronger than the assumptions that we made in previous sections, where we made no separability assumptions at all.

[30] It can be shown that r(p) has the following mathematical properties: r(p) is a nonnegative, nondecreasing, convex and positively linearly homogeneous function for strictly positive p vectors; see Diewert (1974b) or Samuelson and Swamy (1974). A function r(p) is *convex* if for every strictly positive p^1 and p^2 and number λ such that $0 \le \lambda \le 1$, $r(\lambda p^1+(1-\lambda)p^2) \le \lambda r(p^1)+(1-\lambda)r(p^2)$. A function r(p) is *positively linearly homogeneous* if for every positive vector p and positive number λ, we have $r(\lambda p) = \lambda r(p)$.

[31] For additional material on revenue and factor requirements functions, see Diewert (1974b).

Now recall the output revenue maximization problem defined by (1) above. Using the factor requirements function defined by (15) in place of the period t production possibilities set S^t, this revenue maximization problem can be rewritten as follows:

$$
\begin{aligned}
(18)\quad R^t(p,v) &= \max_q \{\textstyle\sum_{n=1}^N p_nq_n : v_1 = G^t(f(q),v_2,\ldots,v_{M+K})\} \\
&= \max_{q,\,Q} \{\textstyle\sum_{n=1}^N p_nq_n : v_1 = G^t(Q,v_2,\ldots,v_{M+K}) \; ; \; Q = f(q)\} \\
&= \max_Q \; \{r(p)Q : v_1 = G^t(Q,v_2,\ldots,v_{M+K})\}
\end{aligned}
$$

where the last equality follows using (17). Now make assumptions (3); i.e., that the observed period t output vector q^t solves the period t revenue maximization problems, which are given by (18) under our separability assumptions (15), with $(p,v) = (p^t,v^t)$ for t = 0,1. Using (18), the following equalities result:

$$
\begin{aligned}
(19)\quad Q^t &= f(q^t) \quad ; \; t = 0,1; \\
(20)\quad R^t(p^t,v^t) &= r(p^t)Q^t \; ; \; t = 0,1.
\end{aligned}
$$

Consider the following revenue maximization problem which uses the period 0 technology, the period 1 output price vector p^1 and the period 0 vector of inputs v^0 :

$$
\begin{aligned}
(21)\quad R^0(p^1,v^0) &= \max_{q,\,Q} \{\textstyle\sum_{n=1}^N p_n^{\,1}q_n : \; v_1^{\,0} = G^0(Q, v_2^{\,0},v_3^{\,0},\ldots,v_{M+K}^{\,0}) \; ; \; Q = f(q)\} \\
&= \max_{q,\,Q} \{\textstyle\sum_{n=1}^N p_n^{\,1}q_n : \; v_1^{\,0} = G^0(Q^0, v_2^{\,0},v_3^{\,0},\ldots,v_{M+K}^{\,0}) \; ; \; Q^0 = f(q)\}
\end{aligned}
$$

since Q^0 will be the only Q that satisfies the constraint $v_1^{\,0} = G^0(Q, v_2^{\,0},v_3^{\,0},\ldots,v_{M+K}^{\,0})$

$$
\begin{aligned}
&= \max_q \{\textstyle\sum_{n=1}^N p_n^{\,1}q_n : Q^0 = f(q)\} \\
&= r(p^1)Q^0 \quad \text{using (17) with } p = p^1 \text{ and } Q = Q^0.
\end{aligned}
$$

Now using the first equality in (20) and the last equality in (21) in order to evaluate *the base period version of the theoretical output price index*, $P^0(p^0,p^1,v^0)$, defined above in (4):

$$
\begin{aligned}
(22)\quad P^0(p^0,p^1,v^0) &\equiv R^0(p^1,v^0)/R^0(p^0,v^0) \\
&= r(p^1)Q^0 / r(p^0)Q^0 \\
&= r(p^1)/r(p^0).
\end{aligned}
$$

Note that the base period output price index $P^0(p^0,p^1,v^0)$ no longer depends on the base period input vector v^0; it is now simply a ratio of unit revenue functions evaluated at the period 1 prices p^1 in the numerator and at the period 0 prices p^0 in the denominator. This is the simplification that the separability assumptions on the technologies for the two periods imply.

Using the same technique of proof that was used to establish (21), it can be shown that under the separability assumptions (15),:

$$(23)\quad R^1(p^0,v^1) = r(p^0)Q^1.$$

Now the second equality in (20) and (23) can be used in order to evaluate *the current period version of the theoretical output price index* $P^1(p^0,p^1,v^1)$ defined above in (5):

$$(24)\quad P^1(p^0,p^1,v^1) \equiv R^1(p^1,v^1)/R^1(p^0,v^1) = r(p^1)Q^1 / r(p^0)Q^1 = r(p^1)/r(p^0).$$

Again, the current period output price index $P^1(p^0,p^1,v^1)$ no longer depends on the current period input vector v^1; it is again the ratio of unit revenue functions evaluated at the period 1 prices p^1 in the numerator and at the period 0 prices p^0 in the denominator.

Note that under our present homogeneous weak separability assumptions, both theoretical output price indexes defined in (4) and (5) collapse down to the same thing, the ratio of unit revenues pertaining to the two periods under consideration, $r(p^1)/r(p^0)$.[32]

Under the separability assumptions (15) on the establishment technologies for periods 0 and 1, (3), (19) and (20) can be rewritten in order to obtain the following decompositions for establishment revenues in periods 0 and 1:

$$(25)\quad R^0(p^0,v^0) = \sum_{n=1}^{N} p_n^0 q_n^0 = r(p^0)f(q^0)\ ;$$
$$(26)\quad R^1(p^1,v^1) = \sum_{n=1}^{N} p_n^1 q_n^1 = r(p^1)f(q^1).$$

The ratio of unit revenues, $r(p^1)/r(p^0)$, has already been identified as the economic output price index under our separability assumptions, (15), so if the ratio of establishment revenues in period 1 to revenues in period 0, $\sum_{n=1}^{N} p_n^1 q_n^1/\sum_{n=1}^{N} p_n^0 q_n^0$, is divided by the output price index, the corresponding *implicit output quantity index*, $Q(p^0,p^1,q^0,q^1)$ is obtained:

$$(27)\quad Q(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^{N} p_n^1 q_n^1/\sum_{n=1}^{N} p_n^0 q_n^0]/[r(p^1)/r(p^0)] = f(q^1)/f(q^0).$$

Thus under the separability assumptions, the economic output quantity index is found to be equal to $f(q^1)/f(q^0)$.[33]

Now a position has been reached to apply the theory of exact index numbers. In the following subsections, some specific assumptions will be made about the functional form for the unit revenue function $r(p)$ or the output aggregator function $f(q)$ and these specific assumptions will enable price index number formulae that are exactly equal to the theoretical output price index, $r(p^1)/r(p^0)$, to be determined. However, before this, it is necessary to develop the mathematics of the revenue maximization problems for periods 0 and 1 in a bit more detail. This is done in the next section.

Problem

[32] The separability assumptions (15) play the same role in the economic theory of output price indexes as the assumption of homothetic preferences does in the economic theory of cost of living indexes.

[33] Note that under the separability assumptions (15), the output price index defined by (24) simplifies to the unit revenue function ratio $r(p^1)/r(p^0)$ which depends *only* on output prices (and not quantities of inputs) and the corresponding quantity index is $f(q^1)/f(q^0)$ which depends *only* on quantities of outputs produced during the two periods under consideration.

2. Assume that $f(q)$ is increasing and continuous in q for $q \geq 0_N$. Show that the unit revenue function $r(p)$ defined by (16) above has the following properties: $r(p)$ is defined over the set of strictly positive price vectors $p >> 0_N$ and is a (i) nonnegative, (ii) nondecreasing, (iii) convex and (iv) positively linearly homogeneous function.

7. The Mathematics of the Revenue Maximization Problem

In subsequent material, two additional results from economic theory will be needed: Wold's Identity and Hotelling's Lemma. These two results follow from the assumption that the establishment is maximizing revenue during the two periods under consideration subject to the constraints of technology. Wold's Identity tells us that the partial derivative of an output aggregator function with respect to an output quantity is proportional to its output price while Hotelling's Lemma tells us that the partial derivative of a unit revenue function with respect to an output price is proportional to the equilibrium output quantity. These two results enable specific functional forms for the aggregator function $f(q)$ or for the unit revenue function $r(p)$ to be related to bilateral price and quantity indexes, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, that depend on the observable price and quantity vectors pertaining to the two periods under consideration. In particular, Wold's Identity, (29), is used to establish (39) and (51) below while Hotelling's Lemma, (35), is used to establish (45) and (56) below.

Wold's (1944; 69-71) (1953; 145) *Identity* is the following result[34]. Assume that the establishment technologies satisfy the separability assumptions (15) for periods 0 and 1. Assume in addition that the observed period t output vector q^t solves the period t revenue maximization problems, which are defined by (18) under our separability assumptions, with $(p,v) = (p^t,v^t)$ for $t = 0,1$. Finally, assume that the output aggregator function $f(q)$ is differentiable with respect to the components of q at the points q^0 and q^1. Then it can be shown[35] that the following equations hold:

(28) $p_n^t / \sum_{k=1}^N p_k^t q_k^t = [\partial f(q^t)/\partial q_n] / \sum_{n=1}^N p_k q_k^t \, \partial f(q^t)/\partial q_k$; $t = 0,1$; $n = 1,\ldots,N$

where $\partial f(q^t)/\partial q_n$ denotes the partial derivative of the revenue function f with respect to the nth quantity q_n evaluated at the period t quantity vector q^t.

[34] Actually, Wold derived his result in the context of a consumer utility maximization problem but his result carries over to the present production context.

[35] To prove this, consider the first order necessary conditions for the strictly positive vector q^t to solve the period t revenue maximization problem, $\max_q \{\sum_{n=1}^N p_n^t q_n : f(q_1,\ldots,q_N) = f(q_1^t,\ldots,q_N^t) \equiv Q^t\}$. The necessary conditions of Lagrange for q^t to solve this problem are: $p^t = \lambda^t \nabla f(q^t)$ where λ^t is the optimal Lagrange multiplier and $\nabla f(q^t)$ is the vector of first order partial derivatives of f evaluated at q^t. Now take the inner product of both sides of this equation with respect to the period t quantity vector q^t and solve the resulting equation for λ^t. Substitute this solution back into the vector equation $p^t = \lambda^t \nabla f(q^t)$ and we obtain (28).

Since the output aggregator function f(q) has been assumed to be linearly homogeneous, Wold's Identity (28) simplifies into the following equations which will prove to be very useful:[36]

$$(29)\quad p_n^t/\sum_{n=1}^N p_k^t q_k^t = [\partial f(q^t)/\partial q_n]/f(q^t)\ ; \qquad n = 1,\ldots,N\ ;\ t = 0,1.$$

In words, (29) says that the vector of period t establishment output prices p^t divided by period t establishment revenues $\sum_{n=1}^N p_k^t q_k^t$ is equal to the vector of first order partial derivatives of the establishment output aggregator function $\nabla f(p^t) \equiv [\partial f(q^t)/\partial q_1,...,\partial f(q^t)/\partial q_N]$ divided by the period t output aggregator function $f(q^t)$.

Under assumptions (3) and our separability assumptions (15), q^t solves the following revenue maximization problem:

$$(30)\quad \max_q \{\sum_{n=1}^N p_n^t q_n : f(q_1,...,q_N) = f(q_1^t,...,q_N^t)\} = r(p^t)Q^t\ ; \qquad t = 0,1$$

where $Q^t \equiv f(q^t)$ and the last equality follows using (20). Consider the period t revenue maximization problem defined by (30) above. *Hotelling's* (1932; 594) *Lemma* is the following result. If the unit revenue function $r(p^t)$ is differentiable with respect to the components of the price vector p, then the period t quantity vector q^t is equal to the period t production aggregate Q^t times the vector of first order partial derivatives of the unit revenue function with respect to the components of p evaluated at the period t price vector p^t; i.e.,

$$(31)\quad q_n^t = Q^t \partial r(p^t)/\partial p_n\ ; \qquad n = 1,\ldots,N\ ;\ t = 0,1.$$

To explain why (31) holds, consider the following argument. Because it is being assumed that the observed period t quantity vector q^t solves the revenue maximization problem that corresponds to $r(p^t)Q^t$, then q^t must be feasible for this maximization problem so it is necessary that $f(q^t) = Q^t$. Thus q^t is a feasible solution for the following revenue maximization problem where the general price vector p has replaced the specific period t price vector p^t:

$$(32)\quad r(p^t)Q^t \equiv \max_q \{\sum_{n=1}^N p_n^t q_n : f(q_1,\ldots,q_N) = Q^t\}$$
$$\geq \sum_{n=1}^N p_n^t q_n^t$$

where the inequality follows from the fact that $q^t \equiv (q_1^t,\ldots,q_N^t)$ is a feasible (but usually not optimal) solution for the revenue maximization problem in (32). Now define for each strictly positive price vector p the function g(p) as follows:

$$(33)\quad g(p) \equiv \sum_{n=1}^N p_n q_n^t - r(p)Q^t$$

[36] Differentiate both sides of the equation $f(\lambda q) = \lambda f(q)$ with respect to λ and then evaluate the resulting equation at $\lambda = 1$. The equation $\sum_{n=1}^N f_n(q)q_n = f(q)$ results where $f_n(q) \equiv \partial f(q)/\partial q_n$.

where as usual, $p \equiv (p_1,\ldots,p_N)$. Using (30) and (33), it can be seen that $g(p)$ is maximized (over all strictly positive price vectors p) at $p = p^t$. Thus the first order necessary conditions for maximizing a differentiable function of N variables hold, which simplify to equations (31).

Combining equations (19), (25) and (26), yields the following equations:

$$(34)\quad \sum_{n=1}^{N} p_n^t q_n^t = r(p^t)f(q^t) = r(p^t)Q^t \qquad \text{for } t = 0,1.$$

Combining equations (31) and (34), yields the following system of equations:

$$(35)\quad q_n^t / \sum_{n=1}^{N} p_k^t q_k^t = [\partial r(p^t)/\partial p_n]/r(p^t) ; \qquad n = 1,\ldots,N ; t = 0,1.$$

In words, (35) says that the vector of period t establishment outputs q^t divided by period t establishment revenues $\sum_{k=1}^{N} p_k^t q_k^t$ is equal to the vector of first order partial derivatives of the establishment unit revenue function $\nabla r(p^t) \equiv [\partial r(p^t)/\partial p_1,\ldots,\partial r(p^t)/\partial p_N]$ divided by the period t unit revenue function $r(p^t)$.

Note the symmetry of equations (35) with equations (29). It is these two sets of equations that shall be used in subsequent material.

8. Superlative Indexes: The Fisher Ideal Index

Suppose the producer's output aggregator function has the following functional form:[37]

$$(36)\quad f(q_1,\ldots,q_N) \equiv [\sum_{i=1}^{N} \sum_{k=1}^{N} a_{ik}\, q_i q_k]^{1/2} ; \quad a_{ik} = a_{ki} \quad \text{for all i and k.}$$

Differentiating the $f(q)$ defined by (36) with respect to q_i yields the following equations:

$$\begin{aligned}(37)\quad f_i(q) &= (1/2)\, [\sum_{i=1}^{N} \sum_{k=1}^{N} a_{jk}\, q_j q_k]^{-1/2}\, 2\sum_{k=1}^{N} a_{ik}\, q_k ; && i = 1,\ldots,N \\ &= \sum_{i=1}^{N} a_{ik}\, q_k / f(q) && \text{using (36)}\end{aligned}$$

where $f_i(q) \equiv \partial f(q^t)/\partial q_i$. In order to obtain the first equation in (37), the symmetry conditions, $a_{ik} = a_{ki}$ are needed. Now evaluate the second equation in (37) at the observed period t quantity vector $q^t \equiv (q_1^t,\ldots,q_N^t)$ and divide both sides of the resulting equation by $f(q^t)$. We obtain the following equation:

$$(38)\quad f_i(q^t)/f(q^t) = \sum_{k=1}^{N} a_{ik}\, q_k^t / [f(q^t)]^2 \qquad t = 0,1 ; i = 1,\ldots,N.$$

Assume revenue maximizing behavior for the producer in periods 0 and 1. Since the aggregator function f defined by (36) is linearly homogeneous and differentiable, equations (29) will hold (Wold's Identity). Now recall the definition of the Fisher ideal price index, P_F defined by (9) above. If the period 1 revenues are divided by the period 0

[37] The material in this section is an adaptation of the material presented in section 4 of chapter 4 to the producer context from the consumer context.

revenues and then this value ratio is divided by P_F, then the Fisher ideal quantity index, $Q_{F,}$ results:

$$
\begin{aligned}
(39)\quad Q_F(p^0,p^1,q^0,q^1) &\equiv [\textstyle\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^0q_i^0]/P_F(p^0,p^1,q^0,q^1) \\
&= [\textstyle\sum_{i=1}^N p_i^0q_i^1/\sum_{k=1}^N p_k^0q_k^0]^{1/2}\,[\sum_{i=1}^N p_i^1q_i^1/\sum_{k=1}^N p_k^1q_k^0]^{1/2} \\
&= [\textstyle\sum_{i=1}^N f_i(q^0)q_i^1/f(q^0)]^{1/2}\,[\sum_{i=1}^N p_i^1q_i^1/\sum_{k=1}^N p_k^1q_k^0]^{1/2} && \text{using (29) for } t = 0 \\
&= [\textstyle\sum_{i=1}^N f_i(q^0)q_i^1/f(q^0)]^{1/2}\,/[\sum_{k=1}^N p_k^1q_k^0\,/\sum_{i=1}^N p_i^1q_i^1]^{1/2} \\
&= [\textstyle\sum_{i=1}^N f_i(q^0)q_i^1/f(q^0)]^{1/2}\,/[\sum_{i=1}^N f_i(q^1)q_i^0\,/f(q^1)]^{1/2} && \text{using (29) for } t = 1 \\
&= [\textstyle\sum_{i=1}^N \sum_{k=1}^N a_{ik}\, q_k^0q_i^1/[f(q^0)]^2]^{1/2}\,/[\sum_{i=1}^N \sum_{k=1}^N a_{ik}\, q_k^1q_i^0\,/[f(q^1)]^2]^{1/2} && \text{using (38)} \\
&= [1/[f(q^0)]^2]^{1/2}\,/\,[1/[f(q^1)]^2]^{1/2} && \text{using (36) and canceling terms} \\
&= f(q^1)/f(q^0).
\end{aligned}
$$

Thus under the assumption that the producer engages in revenue maximizing behavior during periods 0 and 1 and has technologies in periods 0 and 1 that satisfy the separability assumptions (15), then the Fisher ideal quantity index Q_F is exactly equal to the true quantity index, $f(q^1)/f(q^0)$.[38]

The price index that corresponds to the Fisher quantity index Q_F using the product test (16) in chapter 1 is the Fisher price index P_F defined by (9) in the present chapter. Let r(p) be the unit revenue function that corresponds to the homogeneous quadratic aggregator function f defined by (36). Then using (25), (26) and (39), it can be seen that

$$(40)\quad P_F(p^0,p^1,q^0,q^1) = r(p^1)/r(p^0).$$

Thus under the assumption that the producer engages in revenue maximizing behavior during periods 0 and 1 and has production technologies that satisfy the separability assumptions (15) during periods 0 and 1,then the Fisher ideal price index P_F is exactly equal to the true price index, $r(p^1)/r(p^0)$.

A twice continuously differentiable function f(q) of N variables $q \equiv (q_1,\ldots,q_N)$ can provide a *second order approximation* to another such function f*(q) around the point q* if the level and all of the first and second order partial derivatives of the two functions coincide at q*. It can be shown[39] that the homogeneous quadratic function f defined by (36) can provide a second order approximation to an arbitrary f* around any (strictly positive) point q* in the class of linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (36) is a *flexible functional form*.[40] Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[41] Equation (39) and the fact that the homogeneous quadratic function f

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[38] For the early history of this result in the consumer context, see Diewert (1976; 184).

[39] See Diewert (1976; 130) and let the parameter r equal 2.

[40] Diewert (1974a; 133) introduced this term to the economics literature.

[41] Fisher (1922; 247) used the term *superlative* to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

defined by (36) is a flexible functional form shows that the Fisher ideal quantity index Q_F is a superlative index number formula. Since the Fisher ideal price index P_F also satisfies (40) where r(p) is the unit revenue function that is generated by the homogeneous quadratic utility function, P_F is also a superlative index number formula.

It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the producer's output aggregator function is the homogeneous quadratic function defined by (36), start with the assumption that the producer's unit revenue function is a homogeneous quadratic.[42] Thus suppose that the producer has the following unit revenue function:

$$(41)\quad r(p_1,\ldots,p_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N b_{ik}\, p_i p_k]^{1/2}$$

where the parameters b_{ik} satisfy the following symmetry conditions:

$$(42)\quad b_{ik} = b_{ki} \quad \text{for all i and k.}$$

Differentiating r(p) defined by (41) with respect to p_i yields the following equations:

$$\begin{aligned}(43)\quad r_i(p) &= (1/2)[\sum_{i=1}^N \sum_{k=1}^N b_{jk}\, p_j p_k]^{-1/2}\, 2\sum_{k=1}^N b_{ik}\, p_k\,; && i = 1,\ldots,N \\ &= \sum_{k=1}^N b_{ik}\, p_k / r(p) && \text{using (41)}\end{aligned}$$

where $r_i(p) \equiv \partial r(p^t)/\partial p_i$. In order to obtain the first equation in (43), it is necessary to use the symmetry conditions, (42). Now evaluate the second equation in (43) at the observed period t price vector $p^t \equiv (p_1{}^t,\ldots,p_N{}^t)$ and divide both sides of the resulting equation by $r(p^t)$. The following equations result:

$$(44)\quad r_i(p^t)/r(p^t) = \sum_{k=1}^K b_{ik}\, p_k{}^t / [r(p^t)]^2 \qquad t = 0,1\,;\ i = 1,\ldots,N.$$

As revenue maximizing behavior is assumed for the producer in periods 0 and 1 and since the unit revenue function r defined by (41) is differentiable, equations (35) will hold (Hotelling's Lemma). Now recall the definition of the Fisher ideal price index, P_F defined by (9) above:

$$\begin{aligned}(45)\quad P_F(p^0,p^1,q^0,q^1) &\equiv [\sum_{i=1}^N p_i{}^1 q_i{}^0/\sum_{k=1}^N p_k{}^0 q_k{}^0]^{1/2}\, [\sum_{i=1}^N p_i{}^1 q_i{}^1/\sum_{k=1}^N p_k{}^0 q_k{}^1]^{1/2} \\ &= [\sum_{i=1}^N p_i{}^1 r_i(p^0)/r(p^0)]^{1/2}\, [\sum_{i=1}^N p_i{}^1 q_i{}^1/\sum_{k=1}^N p_k{}^0 q_k{}^1]^{1/2} && \text{using (35) for t = 0} \\ &= [\sum_{i=1}^N p_i{}^1 r_i(p^0)/r(p^0)]^{1/2} / [\sum_{k=1}^N p_k{}^0 q_k{}^1 / \sum_{i=1}^N p_i{}^1 q_i{}^1]^{1/2} \\ &= [\sum_{i=1}^N p_i{}^1 r_i(p^0)/r(p^0)]^{1/2} / [\sum_{i=1}^N p_i{}^0 r_i(p^1)/r(p^1)]^{1/2} && \text{using (35) for t = 1} \\ &= [\sum_{i=1}^N \sum_{k=1}^N b_{ik}\, p_k{}^0 p_i{}^1/[r(p^0)]^2]^{1/2} / [\sum_{i=1}^N \sum_{k=1}^N b_{ik}\, p_k{}^1 p_i{}^0/[r(p^1)]^2]^{1/2} && \text{using (44)} \\ &= [1/[r(p^0)]^2]^{1/2} / [1/[r(p^1)]^2]^{1/2} && \text{using (42) and canceling terms} \\ &= r(p^1)/r(p^0).\end{aligned}$$

[42] Given the producer's output aggregator function r(p), it is possible to modify a technique in Diewert (1974a; 112) and show that the corresponding output aggregator function f(q) can be defined as follows: for a strictly positive quantity vector q, $f(q) \equiv \max_p \{\sum_{i=1}^N p_i q_i : r(p) = 1\}$. See part (e) of problem 3 below.

Thus under the assumption that the producer engages in revenue maximizing behavior during periods 0 and 1 and has technologies that satisfy the separability assumptions (15) and the functional form for the unit revenue function that corresponds to the output aggregator function f(q) is given by (41), then the Fisher ideal price index P_F is *exactly* equal to the true price index, $r(p^1)/r(p^0)$.[43]

Since the homogeneous quadratic unit revenue function r(p) defined by (41) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $r(p^1)/r(p^0)$ means that P_F is a *superlative index number formula*.[44]

Suppose that the b_{ik} coefficients in (41) satisfy the following restrictions:

$$(46)\quad b_{ik} = b_i b_k \qquad\qquad \text{for } i,k = i,\ldots,N$$

where the N numbers b_i are nonnegative. In this special case of (41), it can be seen that the unit revenue function simplifies as follows:

$$\begin{aligned}(47)\quad r(p_1,\ldots,p_N) &\equiv [\sum_{i=1}^N \sum_{k=1}^N b_i b_k\, p_i p_k]^{1/2} \\ &= [\sum_{i=1}^N b_i p_i \sum_{k=1}^N b_k\, p_k]^{1/2} \\ &= \sum_{i=1}^N b_i p_i\, .\end{aligned}$$

Substituting (47) into Hotelling's Lemma (31) yields the following expressions for the period t quantity vectors, q^t:

$$(48)\quad q_n^{\,t} = Q^t\, \partial r(p^t)/\partial p_n = b_n\, Q^t \qquad\qquad n = 1,\ldots,N\, ;\ t = 0,1.$$

Thus if the producer has the output aggregator function that corresponds to the unit revenue function defined by (41) where the b_{ik} satisfy the restrictions (46), then the period 0 and 1 quantity vectors are equal to a multiple of the vector $b \equiv (b_1,\ldots,b_N)$; i.e., $q^0 = bQ^0$ and $q^1 = bQ^1$. Under these assumptions, the Fisher, Paasche and Laspeyres indexes, P_F, P_P and P_L, *all coincide.* However, the output aggregator function f(q) which corresponds to this unit revenue function is not consistent with normal producer behavior since the output production possibilities set in this case is block shaped and hence the producer will not substitute towards producing more expensive commodities from cheaper commodities if relative prices change going from period 0 to 1.

Problem

3. Suppose r(p) is a (i) nonnegative, (ii) nondecreasing, (iii) convex and (iv) positively linearly homogeneous function for $p \geq 0_N$. The function r(p) can be used to define an

[43] This result was obtained by Diewert (1976; 133-134) in the consumer context; see also section 4 of chapter 4 above.

[44] Note that we have shown that the Fisher index P_F is exact for the output aggregator function defined by (36) as well as the output aggregator function that corresponds to the unit revenue function defined by (41). These two output aggregator functions do not coincide in general. However, if the N by N symmetric matrix A of the a_{ik} has an inverse, then it can readily be shown that the N by N matrix B of the b_{ik} will equal A^{-1}.

output possibilities set, S, that corresponds to the set of nonnegative output vectors $q \geq 0_N$ that can be produced by Q units of aggregate input as follows:

(i) $S(Q) \equiv \{q: p^Tq \leq r(p)Q$ for every $p \geq 0_N$ but $p \neq 0_N$; $q \geq 0_N\}$.

If q^* is strictly positive so that $q^* >> 0_N$ and q^* belongs to the frontier of $S(Q^*)$ for some $Q^* > 0$, then $Q^* = f(q^*)$ where f is the factor requirements function that corresponds to the given unit revenue function r(p). Thus since q^* belongs to the frontier of $S(Q^*)$, using definition (i), it must be the case that

(ii) $p^Tq^* \leq r(p)Q^*$ for every $p \geq 0_N$ but $p \neq 0_N$.

It must also be the case that there exists a $p^* > 0_N$ such that

(iii) $p^{*T}q^* = r(p^*)Q^*$.

(a) Explain why (iii) must hold. Hint: if it does not hold, then (ii) holds with a strict inequality for every $p \geq 0_N$ but $p \neq 0_N$. Under these conditions, can q^* belong to the frontier of $S(Q^*)$ as was assumed?

(b) Show that the following condition is equivalent to (ii):

(iv) $r(p)Q^* \geq 1$ for all $p \geq 0_N$ such that $p^Tq^* = 1$.

Hint: Use the fact that r(p) is homogeneous of degree one and note that we assumed that $q^* >> 0_N$.

(c) Use part (b) to show that the Q^* that corresponds to q^* satisfies the following equation:

(v) $\min_p \{r(p): p \geq 0_N$ and $p^Tq^* = 1\}Q^* = 1$.

Thus if the minimum in (v) is positive, we have the following formula for $f(q^*)$ in terms of the unit revenue function r(p):

(vi) $f(q^*) \equiv Q^* = 1/\min_p \{r(p): p^Tq^* = 1; p \geq 0_N\}$.

(d) Show that the following condition is equivalent to (ii):

(vii) $p^Tq^* \leq Q^*$ for every $p \geq 0_N$ such that $r(p) = 1$.

Hint: Since both sides of (ii) are homogeneous of degree one in the components of the price vector p, we can make a convenient normalization on the prices. In (iv), we normalized the left hand side of (ii); now we normalize the right hand side of (ii).

(e) Use part (d) to show that the Q^* that corresponds to q^* satisfies the following equation:

(viii) $f(q^*) \equiv Q^* = \max_p \{p^T q^*: r(p) = 1; p \geq 0_N\}$.

9. Quadratic Mean of Order r Superlative Indexes

It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indexes $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indexes $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $r(p^1)/r(p^0)$ where the aggregator function f or the unit revenue function r is a flexible functional form.[45] Two families of superlative indexes are defined below.

Suppose that the producer's output aggregator function is the *following quadratic mean of order r aggregator function*:[46]

$$(49)\quad f^r(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N a_{ik}\, q_i^{r/2}\, q_k^{r/2}]^{1/r}$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the aggregator function f^r defined by (49) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when $r = 2$, f^2 equals the homogeneous quadratic function defined by (36) above.

Define the quadratic mean of order r quantity index Q^r by:

$$(50)\quad Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0\, (q_i^1/q_i^0)^{r/2}\}^{1/r}\, \{\sum_{i=1}^N s_i^1\, (q_i^1/q_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t / \sum_{k=1}^N p_k^t q_k$ is the period t revenue share for output i as usual. It can be verified that when $r = 2$, Q^r simplifies into Q_F, the Fisher ideal quantity index.

Using exactly the same techniques as were used in the previous section, it can be shown that Q^r is exact for the aggregator function f^r defined by (49); i.e.,

$$(51)\quad Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Thus under the assumption that the producer engages in revenue maximizing behavior during periods 0 and 1 and has technologies that satisfy assumptions (15) where the output aggregator function f(q) is defined by (49), then the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[47] Since Q^r is exact for f^r and f^r is a flexible functional form, the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative quantity indexes.

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[45] This section is an adaptation of the consumer theory material presented in section 5 of chapter 4 to the producer context.
[46] This terminology is due to Diewert (1976; 129).
[47] See Diewert (1976; 130).

For each quantity index Q^r, the product test[48] can be used in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

$$(52)\quad P^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^N p_i^1 q_i^1/\{\sum_{i=1}^N p_i^0 q_i^0\, Q^r(p^0,p^1,q^0,q^1)\} = r^{r*}(p^1)/r^{r*}(p^0)$$

where r^{r*} is the unit revenue function that corresponds to the aggregator function f^r defined by (49) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

When r = 2, Q^r defined by (50) simplifies to Q_F, the Fisher ideal quantity index and P^{r*} defined by (52) simplifies to P_F, the Fisher ideal price index. When r = 1, Q^r defined by (50) simplifies to:

$$\begin{aligned}(53)\quad Q^1(p^0,p^1,q^0,q^1) &\equiv \{\sum_{i=1}^N s_i^0\, (q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^N s_i^1\, (q_i^1/q_i^0)^{-1/2}\} \\ &= [\sum_{i=1}^N p_i^1 q_i^1/\sum_{i=1}^N p_i^0 q_i^0]\{\sum_{i=1}^N p_i^0 q_i^0(q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^N p_i^1 q_i^1(q_i^1/q_i^0)^{-1/2}\} \\ &= [\sum_{i=1}^N p_i^1 q_i^1/\sum_{i=1}^N p_i^0 q_i^0]\{\sum_{i=1}^N p_i^0(q_i^0 q_i^1)^{1/2}\}/\{\sum_{i=1}^N p_i^1(q_i^0 q_i^1)^{1/2}\} \\ &= [\sum_{i=1}^N p_i^1 q_i^1/\sum_{i=1}^N p_i^0 q_i^0]/\{\sum_{i=1}^N p_i^1(q_i^0 q_i^1)^{1/2}/\sum_{i=1}^N p_i^0(q_i^0 q_i^1)^{1/2}\} \\ &\equiv [\sum_{i=1}^N p_i^1 q_i^1/\sum_{i=1}^N p_i^0 q_i^0]/P_W(p^0,p^1,q^0,q^1)\end{aligned}$$

where P_W is the *Walsh* (1901)(1921) *price index* defined previously by (20) in chapter 3. Thus P^{1*} is equal to P_W, the *Walsh price index*, and hence it is also a superlative price index.

Suppose the producer's unit revenue function is the following quadratic mean of order r unit revenue function:[49]

$$(54)\quad r^r(p_1,\ldots,p_n) \equiv [\sum_{i=1}^N \sum_{k=1}^N b_{ik}\, p_i^{r/2}\, p_k^{r/2}]^{1/r}$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the unit revenue function r^r defined by (54) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when r = 2, r^2 equals the homogeneous quadratic function defined by (41) above.

Define the quadratic mean of order r price index P^r by:

$$(55)\quad P^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0\, (p_i^1/p_i^0)^{r/2}\}^{1/r}\, \{\sum_{i=1}^N s_i^1\, (p_i^1/p_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^N p_k^t q_k^t$ is the period t revenue share for output i as usual. It can be verified that when r = 2, P^r simplifies into P_F, the Fisher ideal price index.

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[48] The product test asks that the product of the price index times the quantity index equal the corresponding value ratio between the two periods under consideration; recall equation (1) in chapter 3.

[49] This terminology is due to Diewert (1976; 130). This functional form was first defined by Denny (1974) as a unit cost function.

Using exactly the same techniques as were used in the previous section, it can be shown that P^r is exact for the unit revenue function r^r defined by (54); i.e.,

(56) $P^r(p^0,p^1,q^0,q^1) = r^r(p^1)/r^r(p^0)$.

Thus under the assumption that the producer engages in revenue maximizing behavior during periods 0 and 1 and has technologies that satisfy assumptions (15) where the output aggregator function f(q) corresponds to the unit revenue function $r^r(p)$ defined by (54), then the quadratic mean of order r price index P^r is *exactly* equal to the true output price index, $r^r(p^1)/r^r(p^0)$.[50] Since P^r is exact for r^r and r^r is a flexible functional form, that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative price indexes.

For each price index P^r, the product test can be used in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

(57) $$Q^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^N p_i^1 q_i^1/\{\sum_{i=1}^N p_i^0 q_i^0\, P^r(p^0,p^1,q^0,q^1)\} = f^{r*}(p^1)/f^{r*}(p^0)$$

where f^{r*} is the aggregator function that corresponds to the unit revenue function r^r defined by (54) above.[51] For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

When r = 2, P^r defined by (55) simplifies to P_F, the Fisher ideal price index and Q^{r*} defined by (57) simplifies to Q_F, the Fisher ideal quantity index. When r = 1, P^r defined by (55) simplifies to:

(58) $$\begin{aligned} P^1(p^0,p^1,q^0,q^1) &\equiv \{\sum_{i=1}^N s_i^0\, (p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^N s_i^1\, (p_i^1/p_i^0)^{-1/2}\} \\ &= [\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^0q_i^0]\{\sum_{i=1}^N p_i^0q_i^0(p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^N p_i^1q_i^1(p_i^1/p_i^0)^{-1/2}\} \\ &= [\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^0q_i^0]\{\sum_{i=1}^N q_i^0(p_i^0p_i^1)^{1/2}\}/\{\sum_{i=1}^N q_i^1(p_i^0p_i^1)^{1/2}\} \\ &= [\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^0q_i^0]/\{\sum_{i=1}^N q_i^1(p_i^0p_i^1)^{1/2}/\sum_{i=1}^N q_i^0(p_i^0p_i^1)^{1/2}\} \\ &= [\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^0q_i^0]/Q_W(p^0,p^1,q^0,q^1) \end{aligned}$$

where Q_W is the *Walsh quantity index* defined previously by (40) in chapter 3. Thus Q^{1*} is equal to Q_W, the Walsh (1901) (1921) quantity index, and hence it is also a superlative quantity index.

Essentially, the economic approach to index number theory provides reasonably strong justifications for the use of the Fisher price index P_F defined by (9), the Törnqvist-Theil price index P_T defined by (10), the implicit quadratic mean of order r price indexes P^{r*} defined by (52) (when r = 1, this index is the Walsh price index defined by (20) in

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[50] See Diewert (1976; 133-134).

[51] The function f^{r*} can be defined by using r^r as follows: $f^{r*}(q) \equiv \max_p \{\sum_{i=1}^N p_iq_i : r^r(p) = 1\}$; see problem 3 above.

chapter 3) and the quadratic mean of order r price indexes P^r defined by (55). It is now necessary to ask if it matters which one of these formula is chosen as "best".

10. The Economic Intermediate Input Price Index for an Establishment

We now turn our attention to the economic theory of the *intermediate input price index* for an establishment. This theory is analogous to the economic theory of the output price index explained in section 3 above but now uses the *joint cost function* or the *conditional cost function* C in place of the revenue function R that was used in section 3. Our approach in this section turns out to analogous to the Konüs (1924) theory for the true cost of living index in consumer theory.

Recall that the set S^t describes what output vectors y can be produced in period t if the establishment has at its disposal the vector of inputs $v \equiv [x,z]$, where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus if $[y,x,z] \in S^t$, then the nonnegative output vector y can be produced by the establishment in period t if it can utilize the nonnegative vector x of intermediate inputs and the nonnegative vector z of primary inputs.

Let $p_x \equiv (p_{x1},\ldots p_{xM})$ denote a positive vector of intermediate input prices that the establishment might face in period t, let y be a nonnegative vector of output targets and let z be a nonnegative vector of primary inputs that the establishment might have available for use during period t. Then $\sum_{m=1}^{M}$ the establishment's *conditional cost function* using period t technology is defined as the solution to the following intermediate input cost minimization problem:

$$(59)\quad C^t(p_x,y,z) \equiv \min_{x} \{\textstyle\sum_{m=1}^{M} p_{xm}x_m : (y,x,z) \in S^t\}.$$

Thus $C^t(p_x,y,z)$ is the minimum intermediate input cost, $\Sigma_m\, p_{xm}x_m$, that the establishment must pay in order to produce the vector of outputs y, given that it faces the vector of intermediate input prices p_x and given that the vector of primary inputs z is available for use, using the period t technology.[52]

In order to make the notation for the intermediate input price index comparable to the notation used in earlier chapters for price and quantity indexes, in the remainder of this section, the intermediate input price vector p_x is replaced by the vector p and the vector of intermediate quantities x is replaced by the vector q. Thus $C^t(p_x,y,z)$ is rewritten as $C^t(p,y,z)$.

The period t conditional cost function C^t can be used to define the economy's *period t technology intermediate input price index* P^t between any two periods, say period 0 and period 1, as follows:

$$(60)\quad P^t(p^0,p^1,y,z) = C^t(p^1,y,z)/C^t(p^0,y,z)$$

[52] See McFadden (1978) for the mathematical properties of a conditional cost function. Alternatively, we note that $-C^t(p_x,y,z)$ has the same mathematical properties as the revenue function R^t defined earlier in this chapter.

where p^0 and p^1 are the vectors of intermediate input prices that the establishment faces in periods 0 and 1 respectively, y is a reference vector of outputs that the establishment must produce and z is a reference vector of primary inputs.[53] If M = 1 so that there is only one intermediate input that the establishment uses, then it can be shown that the intermediate input price index collapses down to the single intermediate input price relative between periods 0 and 1, p_1^1/p_1^0. In the general case, note that the intermediate input price index defined by (60) is a ratio of hypothetical intermediate input costs that the establishment must pay in order to produce the vector of outputs y, given that it has the period t technology and the vector of primary inputs v to work with. The numerator in (60) is the minimum intermediate input cost that the establishment could attain if it faced the intermediate input prices of period 1, p^1, while the denominator in (60) is the minimum intermediate input cost that the establishment could attain if it faced the output prices of period 0, p^0. Note that all variables in the numerator and denominator of (60) are held constant except the vectors of intermediate input prices.

As was the case with the theory of the output price index, there are a wide variety of price indexes of the form (60) depending on which (t,y,z) reference vector is chosen; (the reference technology is indexed by t, the reference output vector is indexed by y and the reference primary input vector is indexed by z). As in the theory of the output price index, two special cases of the general definition of the intermediate input price index (60) are of interest: (i) $P^0(p^0,p^1,y^0,z^0)$ which uses the period 0 technology set, the output vector y^0 that was actually produced in period 0 and the primary input vector z^0 that was used in period 0 and (ii) $P^1(p^0,p^1,y^1,z^1)$ which uses the period 1 technology set, the output vector y^1 that was actually produced in period 1 and the primary input vector z^1 that was used in period 1. Let q^0 and q^1 be the observed intermediate input vectors for the establishment in periods 0 and 1 respectively. If there is cost minimizing behavior on the part of the producer in periods 0 and 1, then the observed intermediate input cost in periods 0 and 1 should be equal to $C^0(p^0,y^0,z^0)$ and $C^1(p^1,y^1,z^1)$ respectively; i.e., the following equalities should hold:

(61) $C^0(p^0,y^0,z^0) = \sum_{m=1}^M p_m^0 q_m^0$ and $C^1(p^1,y^1,z^1) = \sum_{m=1}^M p_m^1 q_m^1$.

Under these cost minimizing assumptions, adapt the arguments of Fisher and Shell (1972; 57-58) and Archibald (1977; 66) can again be adapted to show that the two theoretical indexes, $P^0(p^0, p^1, y^0,z^0)$ and $P^1(p^0,p^1,y^1,z^1)$ described in (i) and (ii) above, satisfy the following inequalities (62) and (63):

(62) $P^0(p^0,p^1,y^0,z^0) \equiv C^0(p^1,y^0,z^0)/C^0(p^0,y^0,z^0)$ using definition (60)
$= C^0(p^1,y^0,z^0)/\sum_{m=1}^M p_m^0 q_m^0$ using assumption (61)
$\leq \sum_{m=1}^M p_m^1 q_m^0 / \sum_{m=1}^M p_m^0 q_m^0$
since q^0 is feasible for the minimization problem which defines

[53] This concept of the intermediate input price index is analogous to the import price index which was defined in Alterman, Diewert and Feenstra (1999). If we omit the vector of primary inputs from (60), then the resulting intermediate input price index reduces to the physical production cost index defined by Court and Lewis (1942-3; 30).

defines $C^0(p^1,y^0,z^0)$ and so $C^0(p^1,y^0,z^0) \leq \sum_{m=1}^{M} p_m^{\;1}q_m^{\;0}$
$\equiv P_L(p^0,p^1,q^0,q^1)$

where P_L is the Laspeyres intermediate input price index. Similarly,:

(63) $P^1(p^0,p^1,y^1,z^1) \equiv C^1(p^1,y^1,z^1)/C^1(p^0,y^1,z^1)$ using definition (60)
$= \sum_{m=1}^{M} p_m^{\;1}q_m^{\;1}/C^1(p^0,y^1,z^1)$ using (61)
$\geq \sum_{m=1}^{M} p_m^{\;1}q_m^{\;1}/\sum_{m=1}^{M} p_m^{\;0}q_m^{\;1}$
since q^1 is feasible for the minimization problem which defines $C^1(p^0,y^1,z^1)$ and so $C^1(p^0,y^1,z^1) \leq \sum_{m=1}^{M} p_m^{\;0}q_m^{\;1}$
$\equiv P_P(p^0,p^1,q^0,q^1)$

where P_P is the Paasche price index. Thus the inequality (62) says that the observable Laspeyres index of intermediate input prices P_L is an *upper bound* to the theoretical intermediate input price index $P^0(p^0,p^1,y^0,z^0)$ and the inequality (63) says that the observable Paasche index of intermediate input prices P_P is a *lower bound* to the theoretical intermediate input price index $P^1(p^0,p^1,y^1,z^1)$. Note that these inequalities are the *reverse* of our earlier inequalities (4) and (5) that was found for the output price index[54] but our new inequalities are analogous to their counterparts in the theory of the true cost of living index.

As was the case in section 4 above, it is possible to define a theoretical intermediate input price index that falls *between* the observable Paasche and Laspeyres intermediate input price indexes. To do this, first define a *hypothetical intermediate input cost function*, $C(p,\alpha)$, that corresponds to the use of an α weighted average of the technology sets S^0 and S^1 for periods 0 and 1 as the reference technology and that uses an α weighted average of the period 0 and period 1 output vectors y^0 and y^1 and primary input vectors z^0 and z^1 as the reference output and primary input vectors:

(64) $C(p,\alpha) \equiv \min_q \{\sum_{m=1}^{M} p_m q_m : [q, (1-\alpha)y^0 + \alpha y^1, (1-\alpha)z^0 + \alpha z^1] \in [(1-\alpha)S^0 + \alpha S^1]\}$.

Thus the intermediate input cost minimization problem in (64) corresponds to the intermediate output target $(1-\alpha)y^0 + \alpha y^1$ and the use of an average of the period 0 and 1 primary input vectors z^0 and z^1 where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α and an "average" is used of the period 0 and period 1 technology sets where the period 0 set gets the weight $1-\alpha$ and the period 1 set gets the weight α, where α is a number between 0 and 1. The new intermediate input cost function defined by (64) can now be used to define the following *family of theoretical intermediate input price indexes*:

(65) $P(p^0,p^1,\alpha) \equiv C(p^1,\alpha)/C(p^0,\alpha)$.

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[54] This is due to the fact that we are now dealing with a cost *minimization* problem instead of a revenue *maximization* problem as before.

Adapting the proof of Diewert (1983; 1060-1061) shows that there exists an α between 0 and 1 such that the theoretical intermediate input price index defined by (64) lies between the observable (in principle) Paasche and Laspeyres intermediate input price indices, P_P and P_L ; i.e., there exists an α such that

(66) $P_L \leq P(p^0,p^1,\alpha) \leq P_P$ or $P_P \leq P(p^0,p^1,\alpha) \leq P_L$.

If the Paasche and Laspeyres indices are numerically close to each other, then (66) tells us that a "true" economic intermediate input price index is fairly well determined and a reasonably close approximation to the "true" index can be found by taking a symmetric average of P_L and P_P such as the geometric average which again leads to Irving Fisher's (1922) ideal price index, P_F defined earlier by (9).

It is worth noting that the above theory of the economic intermediate input price indices was very general; in particular, no restrictive functional form or separability assumptions were made on the technology.

The translog technology assumptions used in section 5 above to justify the use of the Törnqvist Theil output price index as an approximation to a theoretical output price index can be adapted to yield a justification for the use of the Törnqvist Theil intermediate input price index as an approximation to a theoretical intermediate input price index. Recall the definition of the period t conditional intermediate input cost function, $C^t(p_x,y,z)$, defined by (59) above. Replace the vector of intermediate input prices p_x by the vector p and define the N+K vector u as $u \equiv [y,z]$. Now assume that the period t conditional cost function has the following *translog functional form* : for t = 0,1 :

(67) $$\ln C^t(p,u) = \alpha_0^t + \sum_{m=1}^M \alpha_m^t \ln p_m + \sum_{j=1}^{N+K} \beta_j^t \ln u_j + \sum_{m=1}^M \sum_{j=1}^M \alpha_{mj}^t \ln p_m \ln p_j + \sum_{m=1}^M \sum_{n=1}^{N+K} \beta_{mn}^t \ln p_m \ln u_n + (1/2) \sum_{n=1}^{N+K} \sum_{k=1}^{N+K} \gamma_{nk}^t \ln u_n \ln u_k$$

where the α_n^t and the α_n^t coefficients satisfy the following restrictions:

(68) $\alpha_{mj}^t = \alpha_{jm}^t$ for all m,j and for t = 0,1; $\gamma_{nk}^t = \gamma_{kn}^t$ for all k,n and for t = 0,1;

(69) $\sum_{m=1}^M \alpha_m^t = 1$ for t = 0,1;

(70) $\sum_{j=1}^M \alpha_{mj}^t = 0$ for t = 0,1 and m = 1,2,…,M;

(71) $\sum_{m=1}^M \beta_{mn}^t = 0$ for t = 0,1 and n = 1,2,…,N+K.

The restrictions (69)-(71) are necessary to ensure that $C^t(p,u)$ is linearly homogeneous in the components of the intermediate input price vector p (which is a property that a conditional cost function must satisfy). Note that at this stage of our argument the coefficients that characterize the technology in each period (the α's, β's and γ's) are allowed to be completely different in each period.

Adapting again the result in Caves, Christensen and Diewert (1982; 1410) to the present context[55]: if the quadratic price coefficients in (67) are equal across the two periods where an index number comparison (i.e., $\alpha_{mj}^{0} = \alpha_{mj}^{1}$ for all m,j) is being made, then the geometric mean of the economic intermediate input price index that uses period 0 technology, the period 0 output vector y^0, and the period 0 vector of primary inputs z^0, $P^0(p^0,p^1,y^0,z^0)$, and the economic intermediate input price index that uses period 1 technology, the period 1 output vector y^1, and the period 1 primary input vector z^1, $P^1(p^0,p^1,y^1,z^1)$, is *exactly* equal to the Törnqvist intermediate input price index P_T defined by (10) above[56]; i.e.,

(72) $P_T(p^0,p^1,q^0,q^1) = [P^0(p^0,p^1,y^0,z^0)\ P^1(p^0,p^1,y^1,z^1)]^{1/2}$.

As was the case with our previous result (14), the assumptions required for the result (72) seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and our assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic intermediate input price index and this corresponds to a flexible functional form, the Törnqvist intermediate input index number formula is said to be *superlative*.

It is possible to adapt the analysis of the output price index that was developed in sections 8 and 9 above to the intermediate input price index and show that the two families of superlative output price indexes, P^{r*} defined by (52) and P^r defined by (55), are also superlative intermediate input price indexes. However, the details are omitted here since in order to derive these results, rather restrictive separability restrictions are required on the technology of the establishment.[57]

In the following section, the analysis presented in sections 3-6 is modified to provide an economic approach to the value added deflator.

11. The Economic Approach to the Value Added Deflator for an Establishment

Attention is now turned to the economic theory of the *value added deflator* for an establishment. This theory is analogous to the economic theory of the output price index explained in section 3 above but now the *profit function* π is used in place of the revenue function R that was used in section 3.

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[55] See problem 1 above for the method of proof. The Caves, Christensen and Diewert translog exactness result is slightly more general than a similar translog exactness result that was obtained earlier by Diewert and Morrison (1986; 668); Diewert and Morrison assumed that all of the quadratic terms in (67) were equal to each other during the two periods under consideration whereas Caves, Christensen and Diewert assumed only that $\alpha_{mj}^{0} = \alpha_{mj}^{1}$ for all m,j.

[56] Of course, in the present context, output prices are replaced by intermediate input prices and the number of terms in the summation of terms defined by (10) is changed from N to M.

[57] The counterpart to our earlier separability assumption (15) is now: $z_1 = F^t(y,x,z_2,...,z_K) = G^t(y,f(x),z_2,...,z_K)$ for t = 0,1 where the intermediate input aggregator function f is linearly homogeneous and independent of t.

Recall that the set S^t describes what output vectors y can be produced in period t if the establishment has at its disposal the vector of inputs [x,z], where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus if $[y,x,z] \in S^t$, then the nonnegative output vector y can be produced by the establishment in period t if it can utilize the nonnegative vector x of intermediate inputs and the nonnegative vector z of primary inputs.

Let $p_y \equiv (p_{y1},\ldots p_{yN})$ and $p_x \equiv (p_{x1},\ldots p_{xM})$ denote positive vectors of output and intermediate input prices that the establishment might face in period t and let z be a nonnegative vector of primary inputs that the establishment might have available for use during period t. Then the establishment's *(gross) profit function* or *net revenue function* using period t technology is defined as the solution to the following net revenue maximization problem:

$$(73)\quad \pi^t(p_y,p_x,z) \equiv \max_{y,x} \{\textstyle\sum_{n=1}^{N} p_{yn}y_n - \sum_{m=1}^{M} p_{xm}x_m : (y,x,z) \in S^t\}$$

where as usual, $y \equiv [y_1,\ldots,y_N]$ is an output vector and $x \equiv [x_1,\ldots,x_M]$ is an intermediate input vector. Thus $\pi^t(p_y,p_x,z)$ is the maximum output revenue, $\sum_{n=1}^{N} p_{yn}y_n$, less intermediate input cost, $\Sigma_m p_{xm}x_m$, that the establishment could generate, given that it faces the vector of output prices p_y and the vector of intermediate input prices p_x and given that the vector of primary inputs z is available for use, using the period t technology.[58]

In the remainder of this section, the *net output price vector* p is defined as $p \equiv [p_y, p_x]$ and the *net output quantity vector* q is defined as $q \equiv [y, -x]$. Thus all output and intermediate input prices are positive, output quantities are positive but intermediate inputs are indexed with a minus sign. With these definitions, $\pi^t(p_y,p_x,z)$ can be rewritten as $\pi^t(p,z)$.

The period t profit function π^t can be used to define the economy's *period t technology value added deflator* P^t between any two periods, say period 0 and period 1, as follows[59]:

$$(74)\quad P^t(p^0,p^1,z) = \pi^t(p^1,z)/\pi^t(p^0,z)$$

where p^0 and p^1 are the N+M dimensional vectors of net output prices that the establishment faces in periods 0 and 1 and z is a reference vector of primary inputs. Note that all variables in the numerator and denominator of (74) are held constant except the vectors of net output (output and intermediate input) prices.

As was the case with the theory of the output price index, there are a wide variety of price indexes of the form (74) depending on which (t,z) reference vector that is chosen. The

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[58] The profit function π^t has the same mathematical properties as the revenue function R^t.

[59] If there are no intermediate inputs, this concept reduces to Archibald's (1977; 61) fixed input quantity output price index. In the case where there is no technical progress between the two periods, this concept reduces to Diewert's (1980; 455-461) (net) output price deflator. Diewert (1983; 1055) considered the general concept, which allows for technical progress between periods.

analysis follows that of the output price index in section 3. As in the theory of the output price index, interest lies in two special cases of the general definition of the value added deflator (74): a theoretical index which uses the period 0 technology set and the primary input vector z^0 that was used in period 0 and one that uses the period 1 technology set and the primary input vector z^1 that was used in period 1. These two theoretical indexes are defined as follows:

(75) $P^0(p^0,p^1,z^0) = \pi^0(p^1,z^0)/\pi^0(p^0,z^0)$;

(76) $P^1(p^0,p^1,z^1) = \pi^1(p^1,z^1)/\pi^1(p^0,z^1)$

where $p^0 \equiv [p_y^0, p_x^0]$ and $p^1 \equiv [p_y^1, p_x^1]$ are the period 0 and 1 vectors of output and intermediate input prices facing the establishment. The observable Laspeyres index of output and intermediate input prices P_L is shown to be a *lower bound* to $P^0(p^0,p^1,z^0)$ defined by (75) and the observable Paasche index of output and intermediate input prices P_P is an *upper bound* to $P^1(p^0,p^1,z^1)$ defined by (76).[67] These inequalities go in the same direction as the earlier inequalities (4) and (5) that were obtained for the output price index.

As was the case in section 4 above, it is possible to define a theoretical value added deflator that falls *between* the observable Paasche and Laspeyres value added deflators. To do this, a *hypothetical net revenue function*, $\pi(p,\alpha)$, is defined that corresponds to an α weighted average of the period 0 and 1 technology sets and an α weighted average of the primary input vectors z^0 and z^1 is used as the reference primary input vector.

Following the arguments made for the output price index, if the Paasche and Laspeyres indexes are numerically close to each other, then a "true" economic value added deflator is fairly well determined and a reasonably close approximation to the "true" index is a symmetric average of P_L and P_P such as the geometric average which again leads to Irving Fisher's ideal price index.[69]

The translog technology assumptions that were used in section 5 above to justify the use of the Törnqvist Theil output price index as an approximation to a theoretical output price

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[67] In order to derive the counterpart to the inequality (4), we require that the hypothetical or unobserved value added $\sum_{n=1}^{N+M} p_n^0 q_n^1 \equiv \sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1$ be positive. In order to derive the counterpart to the inequality (5), we require that the hypothetical value added $\sum_{n=1}^{N+M} p_n^1 q_n^0 \equiv \sum_{n=1}^{N} p_{yn}^1 y_n^0 - \sum_{m=1}^{M} p_{xm}^1 x_m^0$ be positive. If the periods 0 and 1 are quite distant in time or if there are dramatic changes in output or intermediate input prices between the two periods, it can happen that these hypothetical value added sums are negative. In this case, one can try and use the chain principle in order to break up the large price and quantity changes that occurred between periods 0 and 1 into a series of smaller changes. With smaller changes, there is a better chance that the hypothetical value added series will remain positive. This seems consistent with the advice of Burns (1930; 256) on this topic. Under certain circumstances, Bowley (1922; 4) raised the possibility of a negative nominal value added. Burns (1930; 257) noted that this anomaly will generally disappear as we aggregate across establishments or industries.

[69] Burns (1930; 244-247) noted that the Laspeyres, Paasche and Fisher value added deflators could be used to deflate nominal net output or value added into real measures. Burns (1930; 247) also noted that that a Fisher ideal production aggregate built up as the product of the Laspeyres and Paasche quantity indexes (the "index" method) would give the same answer as deflating the nominal value added ratio by the Fisher price index (the "deflating" method).

index can be adapted to yield a justification for the use of the Törnqvist Theil value added price index as an approximation to a theoretical value added deflator. Recall the definition of the period t net revenue function, $\pi^t(p_y,p_x,z)$, defined by (73) above. All that is required is to replace the vectors of output prices p_y and the vector of intermediate input prices p_x by the vector $p \equiv [p_y,p_x]$ and assume that the period t net revenue function has the *translog functional form.* Following the argument for the output price index, if the quadratic price coefficients are equal across the two periods, Törnqvist value added deflator is exactly equal to the geometric mean of the two theoretical indexes defined by (75) and (76) above. Because the index number formula is *exactly* equal to an underlying *flexible* functional form, the Törnqvist value added deflator formula is *superlative*. As was the case with the output price index the assumptions required for this finding seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and our assumptions are consistent with technological progress occurring between the two periods being compared.

It is possible to adapt the analysis of the output price index that was developed in sections 8 and 9 above to the value added deflator and show that the two families of superlative output price indexes, P^{r*} defined by (52) and P^r defined by (55), are also superlative value added deflators.[70] In order to derive these results, rather restrictive separability restrictions are required on the technology of the establishment.[71]

Attention is now turned to the problems involved in aggregating over establishments in order to form national output, intermediate input and value added deflators.

12. Aggregation over Establishments: The National Output Price Index

Assume now that there are E establishments in the economy. The goal in this section is to obtain a national output price index that compares output prices in period 1 to those in period 0 and aggregates over all of the establishments in the economy (or a sector of the economy).

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[70] The value added aggregator function that corresponds to (49) in section 9 is now $f^r(y,x)$. For this functional form, we must have all quantities positive and hence the prices of the outputs must be taken to be positive and the prices of intermediate inputs must be negative for the exactness result (51) to hold. For the unit net revenue function that now corresponds to (54) in section 9, we must have all prices positive, output quantities positive and intermediate input quantities negative for the exactness result (56) to hold.

[71] The counterpart to our earlier separability assumption (15) is now: $z_1 = F^t(y,x,z_2,...,z_K) = G^t(f(y,x),z_2,...,z_K)$ for t = 0,1 where the output and intermediate input aggregator function f is linearly homogeneous and independent of t. This type of separability assumption was first made by Sims (1969). Under this separability assumption, the family of value added deflators defined by (74) simplifies to $r(p^1)/r(p^0)$ where the *unit net revenue function* is defined by $r(p) \equiv \max_q \{\sum_{n=1}^{N+M} p_n q_n : f(q_1,...,q_{N+M}) = 1\}$. Note that these deflators are independent of quantities. Under this separability assumption, the quantity index that corresponds to this real value added deflator is $f(y^1,x^1)/f(y^0,x^0)$ and thus this index depends *only* on quantities. Sims (1977; 129) emphasizes that if we want our measures of real net output to depend only on the quantity vectors of outputs produced and intermediate inputs used, then it will be necessary to make a separability assumption. Since these separability assumptions are very restrictive from an empirical point of view, we have tried to develop economic approaches to the PPI that do not rely on separability assumptions.

For e = 1,2,...,E, let $p^e \equiv (p_1^e,\ldots p_N^e)$ denote a positive vector of output prices that establishment e might face in period t and let $v^e \equiv [x^e,z^e]$ be a nonnegative vector of inputs that establishment e might have available for use during period t. Denote the period t technology set for establishment e by S^{et}. As in section 3 above, *the revenue function for establishment e* can be defined using the period t technology as follows:

$$(77)\quad R^{et}(p^e,v^e) \equiv \max_q \{\textstyle\sum_{n=1}^N p_n^e q_n : (q,v^e)\in S^{et}\}; \qquad e = 1,\ldots,E\ ;\ t = 0,1.$$

Now define the *national revenue function* $R^t(p^1,\ldots,p^E,v^1,\ldots,v^E)$ using period t technologies as the sum of the period t establishment revenue functions R^{et} defined by (77):

$$(78)\quad R^t(p^1,\ldots,p^E,v^1,\ldots,v^E) \equiv \textstyle\sum_{e=1}^E R^{et}(p^e,v^e).$$

We simplify the notation by defining the national price vector p as $p \equiv [p^1,\ldots,p^E]$ and the national input vector v as $v \equiv [v^1,\ldots,v^E]$. With this new notation, $R^t(p^1,\ldots,p^E,v^1,\ldots,v^E)$ can be written as $R^t(p,v)$. Thus $R^t(p,v)$ is the maximum value of output, $\sum_{e=1}^E \sum_{n=1}^N p_n^e q_n^e$, that all establishments in the economy can produce, given that establishment e faces the vector of output prices p^e and given that the vector of inputs v^e is available for use by establishment e, using the period t technologies.

The period t national revenue function R^t can be used to define the *national output price index* using the period t technologies P^t between any two periods, say period 0 and period 1, as follows:

$$(79)\quad P^t(p^0,p^1,v) = R^t(p^1,v)/R^t(p^0,v)$$

where $p^0 \equiv [p^{10},p^{20},\ldots,p^{E0}]$ and $p^1 \equiv [p^{11},p^{21},\ldots,p^{E1}]$ are the national vectors of output prices that the various establishments face in periods 0 and 1 respectively and $v \equiv [v^1,v^2,\ldots,v^E]$ is a reference vector of intermediate and primary inputs for each establishment in the economy.[60] The numerator in (79) is the maximum revenue that the economy could attain (using inputs v) if establishments faced the output prices of period 1, p^1, while the denominator in (79) is the maximum revenue that establishments could attain (using inputs v) if they faced the output prices of period 0, p^0. Note that all of the variables in the numerator and denominator functions are exactly the same, except that the output price vectors differ.

As was the case of a single establishment studied in section 3 above, there are a wide variety of price indexes of the form (79) depending on which reference technology t and reference input vector v that we choose. Thus there is not a single economic price index of the type defined by (79): there is an entire *family* of indexes.

As usual, interest lies in two special cases of the general definition of the output price index (79): (i) $P^0(p^0,p^1,v^0)$ which uses the period 0 establishment technology sets and the input vector v^0 that was actually used in period 0 and (ii) $P^1(p^0,p^1,v^1)$ which uses the

[60] This concept for an economy wide producer output price index may be found in Diewert (2001).

period 1 establishment technology sets and the input vector v^1 that was actually used in period 1. Let q^{e0} and q^{e1} be the observed output vectors for the establishments in periods 0 and 1 respectively for e = 1,...,E. If there is revenue maximizing behavior on the part of each establishment in periods 0 and 1, then the sum of observed establishment revenues in periods 0 and 1 should be equal to $R^0(p^0,v^0)$ and $R^1(p^1,v^1)$ respectively; i.e., the following equalities should hold:

(80) $R^0(p^0,v^0) = \sum_{e=1}^E \sum_{n=1}^N p_n^{e0}q_n^{e0}$ and $R^1(p^1,v^1) = \sum_{e=1}^E \sum_{n=1}^N p_n^{e1}q_n^{e1}$.

Under these revenue maximizing assumptions, adapting the arguments of Fisher and Shell (1972; 57-58) and Archibald (1977; 66), Diewert (2001) showed that the two theoretical indexes, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$ described in (i) and (ii) above, satisfy the following inequalities (81) and (82):

(81) $P^0(p^0,p^1,v^0) \equiv R^0(p^1,v^0)/R^0(p^0,v^0)$ using definition (79)
$= R^0(p^1,v^0)/ \sum_{e=1}^E \sum_{n=1}^N p_n^{e0}q_n^{e0}$ using (80)
$\geq \sum_{e=1}^E \sum_{n=1}^N p_n^{e1}q_n^{e0}/\sum_{e=1}^E \sum_{n=1}^N p_n^{e0}q_n^{e0}$
since q^{e0} is feasible for the maximization problem which defines $R^{e0}(p^{e1},v^{e0})$ and so $R^{e0}(p^{e1},v^{e0}) \geq \sum_{n=1}^N p_n^{e1}q_n^{e0}$ for e = 1,...,E
$\equiv P_L(p^0,p^1,q^0,q^1)$

where P_L is the Laspeyres output price index, which treats each commodity produced by each establishment as a separate commodity. Similarly:

(82) $P^1(p^0,p^1,v^1) \equiv R^1(p^1,v^1)/R^1(p^0,v^1)$ using definition (79)
$= \sum_{e=1}^E \sum_{n=1}^N p_n^{e1}q_n^{e1}/R^1(p^0,v^1)$ using (80)
$\leq \sum_{e=1}^E \sum_{n=1}^N p_n^{e1}q_n^{e1}/\sum_{e=1}^E \sum_{n=1}^N p_n^{e0}q_n^{e1}$
since q^{e1} is feasible for the maximization problem which defines $R^{e1}(p^{e0},v^{e1})$ and so $R^{e1}(p^{e0},v^{e1}) \geq \sum_{n=1}^N p_n^{e0}q_n^{e1}$ for e = 1,...,E
$\equiv P_P(p^0,p^1,q^0,q^1)$

where P_P is the Paasche output price index, which treats each commodity produced by each establishment as a separate commodity. Thus the inequality (81) says that the observable Laspeyres index of output prices P_L is a *lower bound* to the theoretical national output price index $P^0(p^0,p^1,v^0)$ and the inequality (82) says that the observable Paasche index of output prices P_P is an *upper bound* to the theoretical national output price index $P^1(p^0,p^1,v^1)$.

It is possible to relate the Laspeyres type *national* output price index $P^0(p^0,p^1,v^0)$ to the *individual establishment* Laspeyres type output price indexes $P^{e0}(p^{e0},p^{e1},v^{e0})$ defined as follows:

(83) $P^{e0}(p^{e0},p^{e1},v^{e0}) \equiv R^{e0}(p^{e1},v^{e0})/R^{e0}(p^{e0},v^{e0}) = R^{e0}(p^{e1},v^{e0})/\sum_{n=1}^N p_n^{e0}q_n^{e0}$; e = 1,...,E

where the establishment period 0 technology revenue functions R^{e0} were defined above by (77) and assumptions (80) were used to establish the second set of equalities; i.e., the assumption that each establishment's observed period 0 revenues, $\sum_{n=1}^{N} p_n^{e0} q_n^{e0}$, are equal to the optimal revenues, $R^{e0}(p^{e0},v^{e0})$. Now define *the revenue share of establishment e in national revenue for period 0* as

$$(84)\quad S_e^{\,0} \equiv \sum_{n=1}^{N} p_n^{e0} q_n^{e0} / \sum_{i=1}^{E} \sum_{n=1}^{N} p_n^{i0} q_n^{i0}\,; \qquad e = 1,...,E.$$

Using the definition of the Laspeyres type national output price index $P^0(p^0,p^1,v^0)$, definition (81) for $(t,v) = (0,v^0)$, we have:

$$\begin{aligned}(85)\quad P^0(p^0,p^1,v^0) &\equiv \sum_{e=1}^{E} R^{e0}(p^{e1},v^{e0}) / \sum_{e=1}^{E} R^{e0}(p^{e0},v^{e0}) \\ &= \sum_{e=1}^{E} R^{e0}(p^{e0},v^{e0})\,[R^{e0}(p^{e1},v^{e0})/R^{e0}(p^{e0},v^{e0})] / \sum_{e=1}^{E} R^{e0}(p^{e0},v^{e0}) \\ &= \sum_{e=1}^{E} S_e^{\,0}\,[R^{e0}(p^{e1},v^{e0})/R^{e0}(p^{e0},v^{e0})] && \text{using (84)} \\ &= \sum_{e=1}^{E} S_e^{\,0}\, P^{e0}(p^{e0},p^{e1},v^{e0}) && \text{using (83).}\end{aligned}$$

Thus the *Laspeyres type national output price index* $P^0(p^0,p^1,v^0)$ is equal to a *base period establishment revenue share weighted average of the individual establishment Laspeyres type output price indexes* $P^{e0}(p^{e0},p^{e1},v^{e0})$.

Of course, it is also possible to relate the Paasche type *national* output price index $P^1(p^0,p^1,v^1)$ to the *individual establishment* Paasche type output price indexes $P^{e1}(p^{e0},p^{e1},v^{e1})$ defined as follows:

$$(86)\quad P^{e1}(p^{e0},p^{e1},v^{e1}) \equiv R^{e1}(p^{e1},v^{e1})/R^{e1}(p^{e0},v^{e1}) = \sum_{n=1}^{N} p_n^{e1} q_n^{e1} / R^{e1}(p^{e0},v^{e1})\,; \qquad e = 1,...,E$$

where the establishment period 1 technology revenue functions R^{e1} were defined above by (77) and we used assumptions (80) to establish the second set of equalities; i.e., the assumption that each establishment's observed period 1 revenues, $\sum_{n=1}^{N} p_n^{e1} q_n^{e1}$, are equal to the optimal revenues, $R^{e1}(p^{e1},v^{e1})$. Now define *the revenue share of establishment e in national revenue for period 1* as

$$(87)\quad S_e^{\,1} \equiv \sum_{n=1}^{N} p_n^{e1} q_n^{e1} / \sum_{i=1}^{E} \sum_{n=1}^{N} p_n^{i1} q_n^{i1}\,; \qquad e = 1,...,E.$$

Using the definition of the Paasche type national output price index $P^1(p^0,p^1,v^1)$, definition (82), we have:

$$\begin{aligned}(88)\quad P^1(p^0,p^1,v^1) &\equiv \sum_{e=1}^{E} R^{e1}(p^{e1},v^{e1}) / \sum_{e=1}^{E} R^{e1}(p^{e0},v^{e1}) \\ &= 1/[\sum_{e=1}^{E} R^{e1}(p^{e0},v^{e1}) / \sum_{e=1}^{E} R^{e1}(p^{e1},v^{e1})] \\ &= 1/[\sum_{e=1}^{E} R^{e1}(p^{e1},v^{e1})[R^{e1}(p^{e0},v^{e1})/R^{e1}(p^{e1},v^{e1})] / \sum_{e=1}^{E} R^{e1}(p^{e1},v^{e1}) \\ &= 1/\sum_{e=1}^{E} S_e^{\,1}\,[R^{e1}(p^{e1},v^{e1})/R^{e1}(p^{e0},v^{e1})]^{-1} && \text{using (87)} \\ &= \{\sum_{e=1}^{E} S_e^{\,1}\,[P^{e1}(p^{e0},p^{e1},v^{e1})]^{-1}\}^{-1} && \text{using (83).}\end{aligned}$$

Thus the Paasche type national output price index $P^1(p^0,p^1,v^1)$ is equal to *a period 1 establishment revenue share weighted harmonic average* of the individual establishment Paasche type output price indexes $P^{e1}(p^{e0},p^{e1},v^{e1})$.

As was the case in section 4 above, it is possible to define a national output price index that falls *between* the observable Paasche and Laspeyres national output price indexes. To do this, first a *hypothetical revenue function*, $R^e(p^e,\alpha)$, is defined for each establishment that corresponds to the use of an α weighted average of the technology sets S^{e0} and S^{e1} for periods 0 and 1 as the reference technology and that uses an α weighted average of the period 0 and period 1 input vectors v^{e0} and v^{e1} as the reference input vector:

(89) $R^e(p^e,\alpha) \equiv \max_q \{\sum_{n=1}^N p_n^e q_n : [q, (1-\alpha)v^0 + \alpha v^1] \in [(1-\alpha)S^{e0} + \alpha S^{e1}]\}$; $e = 1,...,E$.

Once the establishment hypothetical revenue functions have been defined by (89), *the intermediate technology national revenue function* $R^t(p^1,...,p^E,v^1,...,v^E)$ can be defined as the sum of the period t individual intermediate technology establishment revenue functions R^e as follows:

(90) $R(p^1,...,p^E,\alpha) \equiv \sum_{e=1}^E R^e(p^e,\alpha)$.

Again, simplify the notation by defining the national price vector p as $p \equiv [p^1,...,p^E]$. With this new notation, $R(p^1,...,p^E,\alpha)$ can be written as $R(p,\alpha)$. Now use the national revenue function defined by (90) in order to define the following family of *theoretical national output price indexes*:

(91) $P(p^0,p^1,\alpha) \equiv R(p^1,\alpha)/R(p^0,\alpha)$.

As usual, the proof of Diewert (1983; 1060-1061) can be adapted to show that there exists an α between 0 and 1 such that a theoretical national output price index defined by (91) lies between the observable (in principle) Paasche and Laspeyres national output price indexes defined in (88) and (85), P_P and P_L ; i.e., there exists an α such that

(92) $P_L \le P(p^0,p^1,\alpha) \le P_P$ or $P_P \le P(p^0,p^1,\alpha) \le P_L$.

If the Paasche and Laspeyres indexes are numerically close to each other, then (92) tells us that a "true" national output price index is fairly well determined and a reasonably close approximation can be found to the "true" index by taking a symmetric average of P_L and P_P such as the geometric average which again leads to Irving Fisher's (1922) ideal price index, P_F defined earlier by (9).

The above theory for the national output price indexes is very general; in particular, no restrictive functional form or separability assumptions were made on the establishment technologies.

The translog technology assumptions that were used in section 5 above to justify the use of the Törnqvist Theil output price index for a single establishment as an approximation to a theoretical output price index for a single establishment can be adapted to yield a justification for the use of a national Törnqvist Theil output price index as an approximation to a theoretical national output price index.

Recall the definition of the national period t national revenue function, $R^t(p,v) \equiv R^t(p^1,...,p^E,v^1,...,v^E)$, defined earlier by (78) above. Assume that the period t national revenue function has the following *translog functional form* : for t = 0,1 :

$$\begin{aligned}(93)\ \ln R^t(p,v) = {} & \alpha_0{}^t + \textstyle\sum_{n=1}^{NE} \alpha_n{}^t \ln p_n + \sum_{m=1}^{(M+K)E} \beta_m{}^t \ln v_m \\ & + (1/2) \textstyle\sum_{n=1}^{NE} \sum_{j=1}^{NE} \alpha_{nj}{}^t \ln p_n \ln p_j + \sum_{n=1}^{NE} \sum_{m=1}^{(M+K)E} \beta_{nm}{}^t \ln p_n \ln v_m \\ & + (1/2) \textstyle\sum_{m=1}^{(M+K)E} \sum_{k=1}^{(M+K)E} \delta_{mk}{}^t \ln v_m \ln v_k\end{aligned}$$

where the $\alpha_n{}^t$ coefficients satisfy the restrictions:

$$(94)\ \textstyle\sum_{n=1}^{NE} \alpha_n{}^t = 1 \quad \text{for } t = 0,1$$

and the $\alpha_{nj}{}^t$ and $\beta_{nm}{}^t$ coefficients satisfy the following restrictions:[61]

$$\begin{aligned}(95)\ & \textstyle\sum_{j=1}^{NE} \alpha_{nj}{}^t = 0 \quad \text{for } t = 0,1 \text{ and } n = 1,2,\ldots,NE; \\ & \textstyle\sum_{n=1}^{NE} \beta_{nm}{}^t = 0 \quad \text{for } t = 0,1 \text{ and } m = 1,2,\ldots,(M+K)E.\end{aligned}$$

Note that the national output price vector p in (93) has dimension equal to NE, the number of outputs times the number of establishments; i.e., $p \equiv [p_1,...,p_N; p_{N+1},...,p_{2N}; ...; p_{(E-1)N+1},...,p_{NE}] = [p_1{}^1,...,p_N{}^1; p_1{}^2,...,p_N{}^2; ...; p_1{}^E,...,p_N{}^E]$. Similarly, the national input vector v in (93) has dimension equal to (M+K)E, the number of intermediate and primary inputs in the economy times the number of establishments.[62] The restrictions (94) and (95) are necessary to ensure that $R^t(p,v)$ is linearly homogeneous in the components of the output price vector p (which is a property that a revenue function must satisfy). Note that at this stage of our argument the coefficients that characterize the technology in each period (the α's, β's and γ's) are allowed to be completely different in each period. We also note that the translog functional form is an example of a *flexible* functional form[63]; i.e., it can approximate an arbitrary technology to the second order.

Define the national revenue share for establishment e and output n for period t as follows:

$$(96)\ \ s_n{}^{et} \equiv p_n{}^{et}q_n{}^{et}/\textstyle\sum_{i=1}^{E} \sum_{j=1}^{N} p_j{}^{it}q_j{}^{it} \,; \qquad n = 1,...,N \,;\ e = 1,...,E \,;\ t = 0,1.$$

Using the above establishment revenue shares and the establishment output price relatives, $p_n{}^{e1}/p_n{}^{e0}$, we can define the logarithm of the *national Törnqvist* (1936)(1937) *Theil* (1967) *output price index* P_T as follows:

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[61]It is also assumed that the symmetry conditions $\alpha_{nj}{}^t = \alpha_{jn}{}^t$ for all n,j and for t = 0,1 and $\gamma_{mk}{}^t = \gamma_{km}{}^t$ for all m,k and for t = 0,1 are satisfied.

[62] It has also been implicitly assumed that each establishment can produce each of the N outputs in the economy and that each establishment uses all M+K inputs in the economy. These restrictive assumptions can readily be relaxed but only at the cost of notational complexity. All that is required is that each establishment produces the same set of outputs in each period.

[63] In fact the assumption that the period t national revenue function $R^t(p,v)$ has the translog functional form defined by (93) may be regarded as an approximation to the true technology since (93) has not imposed any restrictions on the national technology that are implied by the fact that the national revenue function is equal to the sum of the establishment revenue functions.

(97) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{e=1}^{E} \sum_{n=1}^{N} (1/2)(s_n^{e0} + s_n^{e1}) \ln (p_n^{e1}/p_n^{e0})$.

Recall Theil's (1967) weighted stochastic approach to index number theory that was explained in section 4 of Chapter 1. In the present context, the discrete random variable R takes on the NE values for the logarithms of the establishment output price ratios between periods 0 and 1, $\ln (p_n^{e1}/p_n^{e0})$, with probabilities $(1/2)(s_n^{e0} + s_n^{e1})$. Thus the right hand side of (97) can also be interpreted as the *mean* of this distribution of economy wide logarithmic output price relatives.

A result in Caves, Christensen and Diewert (1982; 1410) can be adapted to the present context: if the quadratic price coefficients in (93) are equal across the two periods where we are making an index number comparison (i.e., $\alpha_{ij}^0 = \alpha_{ij}^1$ for all i,j), then the geometric mean of the national output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the national output price index that uses period 1 technology and the period 1 input vector v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by (97) above; i.e.,

(98) $P_T(p^0,p^1,q^0,q^1) = [P^0(p^0,p^1,v^0)\, P^1(p^0,p^1,v^1)]^{1/2}$.

As usual, the assumptions required for this result seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and our assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic output price index and this corresponds to a flexible functional form, we say that the Törnqvist national output price index number formula is *superlative* following the terminology used by Diewert (1976).

There are *four important results* in this section which can be summarized as follows.

Define the *national Laspeyres output price index* as follows:

(99) $P_L(p^0,p^1,q^0,q^1) \equiv \sum_{e=1}^{E} \sum_{n=1}^{N} p_n^{e1}q_n^{e0} / \sum_{e=1}^{E} \sum_{n=1}^{N} p_n^{e0}q_n^{e0}$.

Then this national Laspeyres output price index is a *lower bound* to the economic output price index $P^0(p^0,p^1,v^0) \equiv R^0(p^1,v^0)/R^0(p^0,v^0)$ where the national revenue function $R^0(p,v^0)$ using the period 0 technology and input vector v^0 is defined by (77) and (78).

Define the *national Paasche output price index* as follows:

(100) $P_P(p^0,p^1,q^0,q^1) \equiv \sum_{e=1}^{E} \sum_{n=1}^{N} p_n^{e1}q_n^{e1} / \sum_{e=1}^{E} \sum_{n=1}^{N} p_n^{e0}q_n^{e1}$.

Then this national Paasche output price index is an *upper bound* to the economic output price index $P^1(p^0,p^1,v^1) \equiv R^1(p^1,v^1)/R^1(p^0,v^1)$ where the national revenue function $R^1(p,v^1)$ using the period 1 technology and input vector v^1 is defined by (77) and (78).

Define the *national Fisher output price index* P_F as the square root of the product of the national Laspeyres and Paasche indexes defined above:

(101) $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\ P_P(p^0,p^1,q^0,q^1)]^{1/2}$.

Then usually, the national Fisher output price index will be a good approximation to an economic output price index that is based on a revenue function that uses a technology set and an input vector that is intermediate to the period 0 and 1 technology sets and input vectors.

Under the assumption that the period 0 and 1 national revenue functions have translog functional forms, then the geometric mean of the national output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the national output price index that uses period 1 technology and the period 1 input vector v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by (97) above; i.e., we have the equality (98).

This section concludes with an observation. Economic justifications have been presented for the use of the national Fisher output price index, $P_F(p^0,p^1,q^0,q^1)$ defined by (101) above and for the use of the national Törnqvist output price index $P_T(p^0,p^1,q^0,q^1)$ defined by (97) above. The results in section 3 of chapter 9 above indicate that for “normal” time series data, these two indexes will give virtually the same answer.

13. The National Intermediate Input Price Index

The theory of the intermediate input price index for a single establishment that was developed in section 10 above can be extended to the case where there are E establishments in the economy. The techniques used for this extension are very similar to the techniques used in section 12 above, so it is not necessary to replicate this work here.

The observable national Laspeyres index of intermediate input prices is found to be an *upper bound* to the theoretical national intermediate input price index using period 0 technology and inputs and the observable national Paasche index of intermediate input prices P_P is a *lower bound* to the theoretical national intermediate input price index using period 1 technology and inputs.

As was the case in section 10 above, it is possible to define a theoretical national intermediate input price index that falls *between* the observable Paasche and Laspeyres national intermediate input price indexes. Usually, the *national Fisher intermediate input price index* P_F defined as the square root of the product of the national Laspeyres and Paasche indexes will be a good approximation to this economic intermediate input price index. Such an index is based on a national cost function that uses establishment technology sets, target establishment output vectors and establishment primary input vectors that are intermediate to the period 0 and 1 technology sets, observed output vectors and observed primary input vectors.

The translog technology assumptions that were used in section 10 above to justify he use of the Törnqvist Theil intermediate input price index for a single establishment as an approximation to a theoretical intermediate input price index for a single establishment can be adapted to yield a justification for the use of a national Törnqvist Theil intermediate input price index as an approximation to a theoretical national intermediate input price index.

14. The National Value Added Deflator

In this section, it is the theory of the *value added deflator for a single establishment* developed in section 11 above that is drawn on and extended to the case where there are E establishments in the economy. The techniques that we use for this extension are again very similar to the techniques that we used in section 12 above, except that an establishment net revenue functions π^{et} is used in place of establishment revenue functions R^{et}.

The observable Laspeyres index of net output prices is shown to be a *lower bound* to the theoretical national value added deflator based on period 0 technology and inputs and the observable Paasche index of net output prices is an *upper bound* to the theoretical national value added deflator based on period 1 technology and inputs.

Constructing industry indexes, such as Laspeyres and Paasche, from individual establishment indexes and national indexes from individual industry indexes requires weights. It should be noted that *establishment shares of national value added* are used for national value added deflators whereas *establishment shares of the national value of (gross) outputs produced* were used in section 12 for national output price indexes. Results supporting the use of Fisher's ideal index and the Törnqvist index arise from arguments similar to those presented for the national output price index.

Recall Theil's (1967) weighted stochastic approach to index number theory that was explained in section 4 of Chapter 1. If his approach is adapted to the present context, then the discrete random variable R would take on the (N+M)E values for the logarithms of the establishment net output price relatives between periods 0 and 1, $\ln (p_n^{e1}/p_n^{e0})$, with "probabilities" $(1/2)(s_n^{e0} + s_n^{e1})$. Thus under this interpretation of the stochastic approach, it would appear that the right hand side of the Törnqvist Theil index could be interpreted as the *mean* of this distribution of economy wide logarithmic output and intermediate input price relatives. However, in the present context, this stochastic interpretation for the Törnqvist Theil net output price formula breaks down because the "shares" $(1/2)(s_n^{e0} + s_n^{e1})$ are negative when n corresponds to an intermediate input.

15. Relationships between the Output Price, Intermediate Input Price and Value Added Deflators

Let the vectors of output price, output quantity, intermediate input price and intermediate input price vectors for an establishment[64] in period t be denoted by p_y^t, y^t, p_x^t and x^t respectively for t = 0,1. Suppose a bilateral index number formula P is used to construct an establishment output price index, $P(p_y^0,p_y^1,y^0,y^1)$, an establishment intermediate input price index, $P(p_x^0,p_x^1,x^0,x^1)$, and an establishment value added deflator, $P(p^0,p^1,q^0,q^1)$, where as usual, $p^t \equiv [p_y^t,p_x^t]$ and $q^t \equiv [y^t,-x^t]$ for t = 0,1. Two related questions arise:

[64] Instead of "establishment", we could substitute the words "industry" or "national economy".

- How is the value added deflator related to the output price index and the intermediate input price index?
- How can the output price index and the intermediate input price index be combined in order to obtain a value added deflator?

Answers to the above questions can be obtained if use is made of the two stage aggregation procedure explained in section 3 of chapter 9.

In the present application of the two stage aggregation procedure explained in section 3 of chapter 9, let M = 2 and the price and quantity vectors p^{tm} and q^{tm} that appeared in that chapter are now defined as follows:

(102) $p^{t1} \equiv p_y^{\,t}$; $p^{t2} \equiv p_x^{\,t}$; $q^{t1} \equiv y^t$; $q^{t2} \equiv -x^t$; t = 0,1.

Thus the first group of commodities that is being aggregated in the first stage of aggregation are the outputs y^t of the establishment and the second group of commodities that is being aggregated in $\sum_{n=1}^{N}$ p the first stage of aggregation are (minus) the intermediate inputs $-x^t$ of the establishment.

The base period first stage aggregate prices and quantities, $P_j^{\,0}$ and $Q_j^{\,0}$, that appeared in equation (6) of chapter 9 are now defined as follows:

(103) $P_1^{\,0} = P_2^{\,0} \equiv 1$; $Q_1^{\,0} \equiv \sum_{n=1}^{N} p_{yn}^{\,0} y_n^{\,0}$; $Q_2^{\,0} \equiv -\sum_{m=1}^{M} p_{xm}^{\,0} x_m^{\,0}$.

Note that $Q_1^{\,0}$ is the base period value of outputs produced by the establishment and $Q_2^{\,0}$ is minus the value of intermediate inputs used by the establishment in period 0.

Now use our chosen index number formula to construct an output price index, $P(p_y^{\,0},p_y^{\,1},y^0,y^1)$, and an intermediate input price index, $P(p_x^{\,0},p_x^{\,1},x^0,x^1)$. These two numbers are set equal to the aggregate price of establishment output $P_1^{\,1}$ and the aggregate price of intermediate input $P_2^{\,1}$ in period 1; i.e., the bilateral index number formula P is used in order to form the following counterparts to (7) in chapter 9 above:

(104) $P_1^{\,1} \equiv P(p_y^{\,0},p_y^{\,1},y^0,y^1)$; $P_2^{\,1} \equiv P(p_x^{\,0},p_x^{\,1},x^0,x^1)$.

Finally, the following counterparts to (8) in chapter 9 generate the period 1 output quantity aggregate $Q_1^{\,1}$ and (minus) the period 1 input aggregate $Q_2^{\,1}$:

(105) $Q_1^{\,1} \equiv \sum_{n=1}^{N} p_{yn}^{\,1} y_n^{\,1}/P_1^{\,1} = \sum_{n=1}^{N} p_{yn}^{\,1} y_n^{\,1}/P(p_y^{\,0},p_y^{\,1},y^0,y^1)$;
$Q_2^{\,1} \equiv -\sum_{m=1}^{M} p_{xm}^{\,1} x_m^{\,1}/P_2^{\,1} = -\sum_{m=1}^{M} p_{xm}^{\,1} x_m^{\,1}/P(p_x^{\,0},p_x^{\,1},x^0,x^1)$.

Thus the period 1 output aggregate, $Q_1^{\,1}$, is equal to the value of period 1 production, $\sum_{n=1}^{N} p_{yn}^{\,1} y_n^{\,1}$, divided by the output price index, $P(p_y^{\,0},p_y^{\,1},y^0,y^1)$ and (minus) the period 1 intermediate input aggregate, $Q_2^{\,1}$, is equal to minus the period 1 cost of intermediate inputs, $\sum_{m=1}^{M} p_{xm}^{\,1} x_m^{\,1}$, divided by the intermediate input price index, $P(p_x^{\,0},p_x^{\,1},x^0,x^1)$. Thus the period 1 output and intermediate input quantity aggregates are constructed by deflating period 1 value aggregates by an appropriate price index, which may be considered to be a type of double deflation procedure.

The period 0 and 1 subcomponent price vectors P^0 and P^1 and the period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

(106) $P^0 \equiv [P_1^0,P_2^0]$; $P^1 \equiv [P_1^1,P_2^1]$; $Q^0 \equiv [Q_1^0,Q_2^0]$; $Q^1 \equiv [Q_1^1,Q_2^1]$.

Finally, given the aggregate prices and quantity vectors defined in (106), use may again be made of our chosen bilateral index number formula P and the *two stage value added deflator* for the establishment, $P(P^0,P^1,Q^0,Q^1)$ calculated. The construction of this two stage value added deflator provides an answer to the following question: how can the output price index and the intermediate input price index be combined in order to obtain a value added deflator?

It is now necessary to ask whether the two stage value added deflator that was just constructed, $P(P^0,P^1,Q^0,Q^1)$, using the bilateral index number formula P in both stages of aggregation is equal to the value added deflator that was constructed in a single stage aggregation, $P(p^0,p^1,q^0,q^1)$, using the same index number formula P; i.e., we ask whether

(107) $P(P^0,P^1,Q^0,Q^1) = P(p^0,p^1,q^0,q^1)$.

The answer to this question is *yes*, if the Laspeyres or Paasche price index is used at each stage of aggregation; i.e., if $P = P_L$ or if $P = P_P$. The answer is *no* if a superlative price index is used at each stage of aggregation; i.e., if $P = P_F$ or if $P = P_T$. However, using the results explained in section 3 of chapter 9, the difference between the right and left hand sides of (107) will be very small if the Fisher or Törnqvist Theil formulae, P_F or P_T, are used consistently at each stage of aggregation. Thus using a superlative index number formula to construct output price, intermediate input price and value added deflators comes at the cost of small inconsistencies as prices are aggregated up in two or more stages of aggregation, whereas the Laspeyres and Paasche formulae are *exactly* consistent in aggregation. However, the use of the Laspeyres or Paasche formulae also comes at a cost: these indexes will have an indeterminate amount of substitution bias compared to their theoretical counterparts[65] whereas superlative indexes will be largely free of substitution bias.

Given the importance of Paasche and Laspeyres price indexes in statistical agency practice, it is worth writing out explicitly the value added deflator using the two stage aggregation procedure explained above when these two indexes are used as the basic index number formula. If the Laspeyres formula is used, the two sides of (107) become:

$$\begin{aligned}(108)\quad P_L(p^0,p^1,q^0,q^1) &\equiv [\sum_{n=1}^{N} p_{yn}^1 y_n^0 - \sum_{m=1}^{M} p_{xm}^1 x_m^0]/[\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0] \\ &= s_y^0\{[\sum_{n=1}^{N} p_{yn}^1 y_n^0]/[\sum_{n=1}^{N} p_{yn}^0 y_n^0]\} + s_x^0\{[\sum_{m=1}^{M} p_{xm}^1 x_m^0]/[\sum_{m=1}^{M} p_{xm}^0 x_m^0]\} \\ &= s_y^0\, P_L(p_y^0,p_y^1,y^0,y^1) + s_x^0\, P_L(p_x^0,p_x^1,x^0,x^1)\end{aligned}$$

[65] Recall Figure 1 above which illustrated substitution biases for the Laspeyres and Paasche output price indexes.

where the period 0 output "share" s_y^0 and the period 0 intermediate input "share" s_x^0 are defined as follows:

$$(109)\ s_y^0 \equiv [\sum_{n=1}^N p_{yn}^0 y_n^0]/[\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0] = P_1^0 Q_1^0/[P_1^0 Q_1^0 + P_2^0 Q_2^0] ;$$
$$s_x^0 \equiv [-\sum_{m=1}^M p_{xm}^0 x_m^0]/[\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0] = P_2^0 Q_2^0/[P_1^0 Q_1^0 + P_2^0 Q_2^0].$$

Note that s_y^0 will be greater than 1 and s_x^0 will be negative. Thus (108) says that the Laspeyres value added deflator can be written as a weighted "average" of the Laspeyres output price index, $P_L(p_y^0,p_y^1,y^0,y^1)$, and the Laspeyres intermediate input price index, $P_L(p_x^0,p_x^1,x^0,x^1)$. Although the weights sum to 1, s_x^0 is negative and s_y^0 is greater than 1, so these weights are rather unusual.

There is an analogous two stage decomposition for the Paasche value added deflator:

$$\begin{aligned}(110)\ P_P(p^0,p^1,q^0,q^1) &\equiv [\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1]/[\sum_{n=1}^N p_{yn}^0 y_n^1 - \sum_{m=1}^M p_{xm}^0 x_m^1] \\ &= 1/\{[\sum_{n=1}^N p_{yn}^0 y_n^1 - \sum_{m=1}^M p_{xm}^0 x_m^1]/[\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1]\} \\ &= 1/\{s_y^1([\sum_{n=1}^N p_{yn}^0 y_n^1]/[\sum_{n=1}^N p_{yn}^1 y_n^1]) + s_x^1([\sum_{m=1}^M p_{xm}^0 x_m^1]/[\sum_{n=1}^N p_{xm}^1 x_m^1])\} \\ &= \{s_y^1\,[P_P(p_y^0,p_y^1,y^0,y^1)]^{-1} + s_x^1\,[P_P(p_x^0,p_x^1,x^0,x^1)]^{-1}\}^{-1}\end{aligned}$$

where the period 1 output "share" s_y^1 and the period 1 intermediate input "share" s_x^1 are defined as follows:

$$(111)\ s_y^1 \equiv [\sum_{n=1}^N p_{yn}^1 y_n^1]/[\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1] = P_1^1 Q_1^1/[P_1^1 Q_1^1 + P_2^1 Q_2^1];$$
$$s_x^1 \equiv [-\sum_{m=1}^M p_{xm}^1 x_m^1]/[\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1] = P_2^1 Q_2^1/[P_1^1 Q_1^1 + P_2^1 Q_2^1].$$

Note that s_y^1 will be greater than 1 and s_x^1 will be negative. Thus (110) says that the Paasche value added deflator can be written as a weighted *harmonic* "average" of the Paasche output price index, $P_P(p_y^0,p_y^1,y^0,y^1)$, and the Paasche intermediate input price index, $P_P(p_x^0,p_x^1,x^0,x^1)$.

Obviously, the analysis presented in this section on the relationships between the output price, the intermediate input price and the value added deflator for an establishment can be extended to the industry or national levels.

16. Value Added Deflators and the Double Deflation Method for Constructing Real Value Added

In the previous section, it was shown how the Paasche and Laspeyres value added deflators for an establishment were related to the Paasche and Laspeyres output and intermediate input price indexes for an establishment. In this section, this analysis will be extended a bit to look at the problems involved in using these indexes to deflate nominal values into real values. Having defined a value added deflator $P(p^0,p^1,q^0,q^1)$ using some index number formula, the product test, equation (1) in Chapter 3, can be used in order to define a corresponding quantity index $Q(p^0,p^1,q^0,q^1)$, which can be interpreted as the growth rate for *real value added* going from period 0 to 1; i.e., given P, Q can be defined as follows:

$$(112)\quad Q(p^0,p^1,q^0,q^1) \equiv [V^1/V^0]/P(p^0,p^1,q^0,q^1)$$

where V^t is the nominal establishment value added for period t = 0,1.

If the *Laspeyres value added deflator* $P_L(p^0,p^1,q^0,q^1)$ is used as the price index in (112), the resulting quantity index Q is the *Paasche value added quantity index* Q_P defined as follows:

$$(113)\ Q_P(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1]/[\sum_{n=1}^N p_{yn}^1 y_n^0 - \sum_{m=1}^M p_{xm}^1 x_m^0].$$

If the *Paasche value added deflator* $P_P(p^0,p^1,q^0,q^1)$ is used as the price index in (112), the resulting quantity index Q is the *Laspeyres value added quantity index* Q_L defined as follows:

$$(114)\ Q_L(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N p_{yn}^0 y_n^1 - \sum_{m=1}^M p_{xm}^0 x_m^1]/[\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0]\ .$$

Given a generic value added quantity index, $Q(p^0,p^1,q^0,q^1)$, *real value added in period 1 at the prices of period 0*, rva^1 say, can be defined as the period 0 nominal value added of the establishment escalated by the value added quantity index Q; i.e.,

$$(115)\quad rva^1 \equiv V^0\, Q(p^0,p^1,q^0,q^1) = [\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0]\, Q(p^0,p^1,q^0,q^1).$$

If the Laspeyres value added quantity index $Q_L(p^0,p^1,q^0,q^1)$ defined by (114) above is used as the escalator of nominal value added in (115), the following rather interesting decomposition for the resulting period 1 real value added at period 0 prices is obtained:

$$\begin{aligned}(116)\quad rva^1 &\equiv [\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0]\, Q_L(p^0,p^1,q^0,q^1) \\ &= [\sum_{n=1}^N p_{yn}^0 y_n^1 - \sum_{m=1}^M p_{xm}^0 x_m^1] \qquad\qquad \text{using (114)} \\ &= [\sum_{n=1}^N p_{yn}^0 y_n^0]\{[\sum_{n=1}^N p_{yn}^0 y_n^1]/[\sum_{n=1}^N p_{yn}^0 y_n^0]\} \\ &\qquad - [\sum_{m=1}^M p_{xm}^0 x_m^0]\{[\sum_{m=1}^M p_{xm}^0 x_n^1]/[\sum_{m=1}^M p_{xm}^0 x_m^0]\} \\ &\equiv \sum_{n=1}^N p_{yn}^0 y_n^0\, Q_L(p_y^0,p_y^1,y^0,y^1) - \sum_{m=1}^M p_{xm}^0 x_m^0\, Q_L(p_x^0,p_x^1,x^0,x^1).\end{aligned}$$

Thus period 1 real value added at period 0 prices, rva^1, is defined to be period 0 nominal value added, $\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0$, escalated by the Laspeyres value added quantity index, $Q_L(p^0,p^1,q^0,q^1)$, defined by (114). But the last line of (116) shows that rva^1 is also equal to the period 0 value of production, $\sum_{n=1}^N p_{yn}^0 y_n^0$, escalated by the Laspeyres output quantity index[66], $Q_L(p_y^0,p_y^1,y^0,y^1)$, minus the period 0 intermediate input cost, $\sum_{m=1}^M p_{xm}^0 x_m^0$, escalated by the Laspeyres intermediate input quantity index, $Q_L(p_x^0,p_x^1,x^0,x^1)$.

Using (112) yields the following formula for the Laspeyres value added quantity index, Q_L, in terms of the Paasche value added deflator, P_P:

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[66] The use of the Laspeyres output quantity index can be traced back to Bowley (1921; 203) at least.

(117) $Q_L(p^0,p^1,q^0,q^1) = [V^1/V^0]/P_P(p^0,p^1,q^0,q^1)$.

Now substitute (117) into the first line of (116) in order to obtain the following alternative decomposition for the period 1 real value added at period 0 prices, rva^1:

$$(118)\quad \begin{aligned} rva^1 &\equiv [\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1]/P_P(p^0,p^1,q^0,q^1) \\ &= [\sum_{n=1}^N p_{yn}^0 y_n^1 - \sum_{n=1}^M p_{xm}^0 x_m^1] \qquad \text{using (110)} \\ &= [\sum_{n=1}^N p_{yn}^1 y_n^1]/\{[\sum_{n=1}^N p_{yn}^1 y_n^1]/[\sum_{n=1}^N p_{yn}^0 y_n^1]\} \\ &\qquad - [\sum_{m=1}^M p_{xm}^1 x_m^1]/\{[\sum_{m=1}^M p_{xm}^1 x_n^1]/[\sum_{n=1}^N p_{xm}^0 x_m^1]\} \\ &= \sum_{n=1}^N p_{yn}^1 y_n^1/P_P(p_y^0,p_y^1,y^0,y^1) - \sum_{m=1}^M p_{xm}^1 x_m^1/P_P(p_x^0,p_x^1,x^0,x^1). \end{aligned}$$

Thus period 1 real value added at period 0 prices, rva^1, is equal to be period 1 nominal value added, $\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1$, deflated by the Paasche value added deflator, $P_P(p^0,p^1,q^0,q^1)$, defined by (110). But the last line of (118) shows that rva^1 is also equal to the period 1 value of production, $\sum_{n=1}^N p_{yn}^1 y_n^1$, deflated by the Paasche output price index, $P_P(p_y^0,p_y^1,y^0,y^1)$, minus the period 1 intermediate input cost, $\sum_{m=1}^M p_{xm}^1 x_m^1$, deflated by the Paasche intermediate input price index, $P_P(p_x^0,p_x^1,x^0,x^1)$. Thus the use of the Paasche value added deflator leads to a measure of period 1 real value added at period 0 prices, rva^1, that is equal to period 1 deflated output minus period 1 deflated intermediate input and hence this method for constructing a real value added measure is called the *double deflation method*.[67]

There is a less well known method of double deflation that reverses the above roles of Paasche and Laspeyres indices, a method which now be explained. Instead of expressing real value added in period 1 at the prices of period 0, it is also possible to define real value added in period 0 at the prices of period 1, rva^0. Using this methodology, given a generic value added quantity index, the counterpart to (115) is:

$$(119)\quad rva^0 \equiv V^1/Q(p^0,p^1,q^0,q^1) = [\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{m=1}^M p_{xm}^1 x_m^1]/Q(p^0,p^1,q^0,q^1).$$

Thus to obtain period 0 real value added at the prices of period 1, rva^0, take the nominal period 1 value added, V^1, and deflate it by the value added quantity index, $Q(p^0,p^1,q^0,q^1)$.

If the Paasche value added quantity index $Q_P(p^0,p^1,q^0,q^1)$ defined by (113) above is used as the deflator of nominal value added in (119), the following interesting decomposition for the resulting period 0 real value added at period 1 prices is obtained:

$$(120)\quad \begin{aligned} rva^0 &\equiv [\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{n=1}^N p_{xm}^1 x_m^1]/Q_P(p^0,p^1,q^0,q^1) \\ &= [\sum_{n=1}^N p_{yn}^1 y_n^0 - \sum_{n=1}^N p_{xm}^1 x_m^0] \qquad \text{using (113)} \\ &= [\sum_{n=1}^N p_{yn}^1 y_n^1]/\{[\sum_{n=1}^N p_{yn}^1 y_n^1]/[\sum_{n=1}^N p_{yn}^1 y_n^0]\} \\ &\qquad - [\sum_{m=1}^M p_{xm}^1 x_m^1]/\{[\sum_{m=1}^M p_{xm}^1 x_m^1]/[\sum_{m=1}^M p_{xm}^1 x_m^0]\} \\ &\equiv \sum_{n=1}^N p_{yn}^1 y_n^1/Q_P(p_y^0,p_y^1,y^0,y^1) - \sum_{m=1}^M p_{xm}^1 x_m^1/Q_P(p_x^0,p_x^1,x^0,x^1). \end{aligned}$$

[67] See Schreyer (2001; 32). It should be mentioned that there is a great deal of useful material in this book that will be of interest to price statisticians

Thus period 0 real value added at period 1 prices, rva^0, is defined to be period 1 nominal value added, $[\sum_{n=1}^N p_{yn}^1 y_n^1 - \sum_{n=1}^N p_{xm}^1 x_m^1$, deflated by the Paasche value added quantity index, $Q_P(p^0,p^1,q^0,q^1)$, defined by (113). But the last line of (120) shows that rva^0 is also equal to the period 1 value of production, $\sum_{n=1}^N p_{yn}^1 y_n^1$, deflated by the Paasche output quantity index, $Q_P(p_y^0,p_y^1,y^0,y^1)$, minus the period 1 intermediate input cost, $\sum_{m=1}^M p_{xm}^1 x_m^1$, deflated by the Paasche intermediate input quantity index, $Q_P(p_x^0,p_x^1,x^0,x^1)$.

Using (112) yields the following formula for the Paasche value added quantity index, Q_P, in terms of the Laspeyres value added deflator, P_L:

$$(121)\quad Q_P(p^0,p^1,q^0,q^1) = [V^1/V^0]/P_L(p^0,p^1,q^0,q^1).$$

Now substitute (121) into the first line of (120) in order to obtain the following alternative decomposition for the period 0 real value added at period 1 prices, rva^0:

$$\begin{aligned}(122)\quad rva^0 &\equiv [\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0] P_L(p^0,p^1,q^0,q^1) \\ &= [\sum_{n=1}^N p_{yn}^1 y_n^0 - \sum_{n=1}^N p_{xm}^1 x_m^0] \qquad \text{using (108)} \\ &= [\sum_{n=1}^N p_{yn}^0 y_n^0]\{[\sum_{n=1}^N p_{yn}^1 y_n^0]/[\sum_{n=1}^N p_{yn}^0 y_n^0]\} \\ &\qquad - [\sum_{m=1}^M p_{xm}^0 x_m^0]\{[\sum_{m=1}^M p_{xm}^1 x_n^0]/[\sum_{m=1}^M p_{xm}^0 x_m^0]\} \\ &= \sum_{n=1}^N p_{yn}^0 y_n^0\, P_L(p_y^0,p_y^1,y^0,y^1) - \sum_{m=1}^M p_{xm}^0 x_m^0\, P_L(p_x^0,p_x^1,x^0,x^1).\end{aligned}$$

Thus period 0 real value added at period 1 prices, rva^0, is equal to be period 0 nominal value added, $\sum_{n=1}^N p_{yn}^0 y_n^0 - \sum_{m=1}^M p_{xm}^0 x_m^0$, escalated by the Laspeyres value added deflator, $P_L(p^0,p^1,q^0,q^1)$, defined by (108). But the last line of (122) shows that rva^0 is also equal to the period 0 value of production, $\sum_{n=1}^N p_{yn}^0 y_n^0$, escalated by the Laspeyres output price index, $P_L(p_y^0,p_y^1,y^0,y^1)$, minus the period 0 intermediate input cost, $\sum_{m=1}^M p_{xm}^0 x_m^0$, escalated by the Laspeyres intermediate input price index, $P_L(p_x^0,p_x^1,x^0,x^1)$.[68]

17. The Aggregation of Establishment Deflators into a National Value Added Deflator

Once establishment value added deflators have been constructed for each establishment, there remains the problem of aggregating up these deflators into an industry or regional or national value added deflator. The national aggregation problem is considered in this section but the same logic will apply to the regional and industry aggregation problems.[69] Note that in section 14 above, we also considered how the national value added deflator could be constructed but by aggregating together the national output price index with the national intermediate input price index. In this section, we assume that we have constructed establishment (or industry) value added deflators and we consider how these industry value added deflators can be aggregated into a national value added deflator. Thus the method of two stage aggregation considered in this section is different from the method of two stage aggregation considered in section 14.

[68] This method for constructing real value added measures was used by Phillips (1961; 320).

[69] The algebra developed below can also be applied to the problem of aggregating establishment or industry output or intermediate input price indexes into national output or intermediate input price indexes.

Let the vectors of output price, output quantity, intermediate input price and intermediate input price vectors for an establishment e in period t be denoted by p_y^{et}, y^{et}, p_x^{et} and x^{et} respectively for t = 0,1 and e = 1,...,E. As usual, the net price and net quantity vectors for establishment e in period t are defined as as $p^{et} \equiv [p_y^{et},p_x^{et}]$ and $q^{et} \equiv [y^{et},-x^{et}]$ for t = 0,1 and e = 1,...,E. Suppose that a bilateral index number formula P is used to construct a value added deflator, $P(p^{e0},p^{e1},q^{e0},q^{e1})$, for establishment e where e = 1,...,E. Our problem is to somehow aggregate up these establishment indexes into a national value added deflator.

The two stage aggregation procedure explained in section 3 of chapter 9 above is used to do this aggregation. The first stage of the aggregation of price and quantity vectors is for the establishment net output price vectors, p^{et}, and the establishment net output quantity vectors, q^{et}. These establishment price and quantity vectors are combined into national price and quantity vectors, p^t and q^t, as follows:

$$(123)\quad p^t = (p^{1t}, p^{2t}, \ldots ,p^{Et}) \; ; \quad q^t = (q^{1t}, q^{1t}, \ldots ,q^{Et}) \quad ; \quad t = 0,1.$$

For each establishment e, its aggregate price of value added P_e^0 in the base period is set equal to 1 and the corresponding establishment e base period quantity of value added Q_e^0 is defined as the establishment's period 0 value added; i.e.,:

$$(124)\quad P_e^0 \equiv 1 \; ; \; Q_e^0 \equiv \textstyle\sum_{i=1}^{N+M} p_i^{e0} q_i^{e0} \qquad \text{for } e= 1,2,\ldots,E.$$

Now the chosen price index formula P is used in order to construct a period 1 price for the price of value added for each establishment e, say P_e^1 for e = 1,2,…,E:

$$(125)\quad P_e^1 \equiv P(p^{e0},p^{e1},q^{e0},q^{e1}) \qquad\qquad \text{for } e = 1,2,\ldots E.$$

Once the period 1 prices for the E establishments have been defined by (125), then corresponding establishment e period 1 quantities Q_e^1 can be defined by deflating the period 1 establishment values $\sum_{i=1}^{N+M} p_i^{e1} q_i^{e1}$ by the prices P_e^1 defined by (125); i.e.,:

$$(126)\quad Q_e^1 \equiv \textstyle\sum_{i=1}^{N+M} p_i^{e1} q_i^{e1} / P_e^1 \qquad\qquad \text{for } e = 1,2,\ldots,E.$$

The aggregate establishment price and quantity vectors for each period t = 0,1 can be defined using equations (124) to (126) above. Thus the period 0 and 1 establishment value added price vectors P^0 and P^1 are defined as follows:

$$(127)\quad P^0 = (P_1^0, P_2^0,\ldots,P_E^0) \equiv 1_E \; ; \; P^1 = (P_1^1, P_2^1,\ldots,P_E^1)$$

where 1_E denotes a vector of ones of dimension E and the components of P^1 are defined by (125). The period 0 and 1 establishment value added quantity vectors Q^0 and Q^1 are defined as:

$$(128)\quad Q^0 = (Q_1^0, Q_2^0,\ldots,Q_E^0) \; ; \; Q^1 = (Q_1^1, Q_2^1,\ldots,Q_E^1)$$

where the components of Q^0 are defined in (124) and the components of Q^1 are defined by (126). The price and quantity vectors in (127) and (128) represent the results of the first stage aggregation (over commodities within an establishment). These vectors can now be used as inputs into the second stage aggregation problem (which aggregates over establishments); i.e., our chosen price index formula can be applied using the information in (127) and (128) as inputs into the index number formula. The resulting two stage aggregation national value added deflator is $P(P^0,P^1,Q^0,Q^1)$. We now ask whether this two stage index equals the corresponding single stage index $P(p^0,p^1,q^0,q^1)$ that treats each treats each output or intermediate input produced or used by each establishment as a separate commodity, using the same index number formula P; i.e., we ask whether

(129) $P(P^0,P^1,Q^0,Q^1) = P(p^0,p^1,q^0,q^1)$.

If the Laspeyres or Paasche formula is used at each stage of each aggregation, the answer to the above question is *yes*. Thus in particular, the national Laspeyres value added deflator that is constructed in a single stage of aggregation, $P_L(p^0,p^1,q^0,q^1)$, is equal to the two stage Laspeyres value added deflator, $P_L(P^0,P^1,Q^0,Q^1)$, where the Laspeyres formula is used in (125) to construct establishment value added deflators in the first stage of aggregation. If a superlative formula is used at each stage of aggregation, the answer to the above consistency in aggregation question is *no*: the equality (129) using a superlative P will only hold approximately. However, if the Fisher, Walsh or Törnqvist price index formulae are used at each stage of aggregation, the differences between the right and left hand sides of (129) will be very small using normal time series data.

18. The National Value Added Deflator Versus the Final Demand Deflator

In this section, we ask whether there are any relationships between *the national value added deflator* defined in the preceding sections of this Chapter and *the national deflator for final demand expenditures*. In particular, we look for conditions that will imply that the two deflators are exactly equal to each other.

Assume that that the commodity classification for intermediate inputs is exactly the same as the commodity classification for outputs so that in particular, N, the number of outputs, is equal to M, the number of intermediate inputs. This assumption is not restrictive since if N is chosen to be large enough, all produced intermediate inputs can be accommodated in the expanded output classification.[70] With this change in assumptions, the same notation can be used as was used in the previous section. Thus let the vectors of output price, output quantity, intermediate input price and intermediate input price vectors for an establishment e in period t be denoted by p_y^{et}, y^{et}, p_x^{et} and x^{et} respectively for t = 0,1 and e = 1,...,E. As usual, the net price and net quantity vectors for establishment e in period t are defined as $p^{et} \equiv [p_y^{et},p_x^{et}]$ and $q^{et} \equiv [y^{et},-x^{et}]$ for t = 0,1 and e = 1,...,E. Again define the national price and quantity vectors, p^t and q^t, as $p^t \equiv (p^{1t}; p^{2t};$

[70] It is not necessary to assume that each establishment or sector of the economy produces all outputs and uses all intermediate inputs in each of the two periods being compared: all that is required is that if an output is not produced in one period by establishment e, then that output is also not produced in the other period. Similarly, it is required that if an establishment does not use a particular intermediate input in one period, then it also does not use it in the other period.

$\ldots;p^{Et}) = (p_y^{1t},p_x^{1t};p_y^{2t},p_x^{2t};\ldots;p_y^{Et},p_x^{Et})$ and $q^t \equiv (q^{1t}, q^{1t}, \ldots ,q^{Et}) = (y^{1t},-x^{1t};y^{2t},-x^{2t};\ldots;y^{Et},-x^{Et})$ for t = 0,1. As in the previous section, an index number formula P is chosen and the national value added deflator denoted as $P(p^0,p^1,q^0,q^1)$.

Using the above notation, the period t N by E *make matrix* for the economy, Y^t, and the period t N by E *use matrix*, X^t, are defined as follows:

$$(130)\quad Y^t \equiv [y^{1t},y^{2t},\ldots,y^{Et}]\ ;\ X^t \equiv [x^{1t},x^{2t},\ldots,x^{Et}]\ ; \qquad t = 0,1.$$

The *period t final demand vector* for the economy, f^t, can be defined by summing up all of the establishment output vectors y^{et} in the period t make matrix and subtracting all of the establishment intermediate input demand vectors x^{et} in the period t use matrix; i.e., define f^t by:[71]

$$(131)\quad f^t \equiv \sum_{e=1}^{E} y^{et} - \sum_{e=1}^{E} x^{et}\ ; \qquad t = 0,1.$$

Final demand prices are required to match up with the components of the period t final demand quantity vector $f^t = [f_1^t,\ldots,f_N^t]$. The net value of production for commodity n in period t divided by the net deliveries of this commodity to final demand f_n^t is *the period t final demand unit value for commodity n*, p_{fn}^t:

$$(132)\quad p_{fn}^t \equiv [\sum_{e=1}^{E} p_{yn}^{et}y_n^{et} - \sum_{e=1}^{E} p_{xn}^{et}x_n^{et}]/f_n^t\ ; \qquad n = 1,\ldots,N\ ;\ t = 0,1.$$

If (132) is to hold so that production less intermediate input use equals deliveries to final demand for each commodity in period t *and* if the *value* of production less the *value* of intermediate demands is to equal the *value* of final demand for each commodity in period t, then the value added prices defined by (132) must be used as final demand prices.

Define the vector of period t final demand prices as $p_f^t \equiv [p_{f1}^t,p_{f2}^t,\ldots,p_{fN}^t]$ for t = 0,1 where the components p_{fn}^t are defined by (132). The corresponding final demand quantity vectors f^t have already been defined by (131). Hence, a generic price index number formula P can be taken to form the *final demand deflator*, $P(p_f^0,p_f^1,f^0,f^1)$. It is now asked whether this final demand deflator is equal to the national value added deflator $P(p^0,p^1,q^0,q^1)$ defined above by the right hand side of (129); i.e., whether

$$(133)\quad P(p_f^0,p_f^1,f^0,f^1) = P(p^0,p^1,q^0,q^1).$$

Note that the dimensionality of each price and quantity vector that occurs in the left hand side of (133) is N (the number of commodities in our output classification) while the dimensionality of each price and quantity vector that occurs in the right hand side of (133) is 2NE where E is the number of establishments (or industries or sectors that have separate price and quantity vectors for both outputs and intermediate inputs) that are aggregating over.

[71] Components of f^t can be negative if the corresponding commodity is being imported into the economy during period t or if the component corresponds to the change in an inventory item.

The answer to the question asked in the previous paragraph is *no*; in general, it will *not* be the case that the final demand deflator is equal to the national value added deflator.

However, under certain conditions, (133) will hold as an equality. A set of conditions are now developed. The first assumption is that all establishments face the same vector of prices p^t in period t for both the outputs that they produce and for the intermediate inputs that they use; i.e., it is assumed:[72]

$$(134)\quad p_y^{et} = p_x^{et} = p^t \,; \qquad e = 1,...,E \,;\ t = 0,1.$$

If assumptions (134) hold, then it is easy to verify that the vector of period t final demand prices p_f^t defined above by (132) is also equal to the vector of period t basic prices p^t.

If assumptions (134) hold and the price index formula used in both sides of (133) is the *Laspeyres formula*, then it can be verified that (133) will hold as an equality; i.e., the Laspeyres final demand deflator will be equal to the national Laspeyres value added deflator. To see why this is so, use the Laspeyres formula in (133) and for the left hand side index, collect all of the quantity terms both in the numerator and denominator of the index that correspond to the common establishment price for the nth commodity, $p_n^t = p_{yn}^{et} = p_{xn}^{et}$, for e = 1,...,E. Using (131) for t = 0, the resulting sum of collected quantity terms will sum to f_n^0. Since this is true for n = 1,...,N, it can be seen that the left hand side Laspeyres index is equal to the right hand side Laspeyres index.

If assumptions (134) hold and the price index formula used in both sides of (133) is the *Paasche formula*, then it can be verified that (133) will also hold as an equality; i.e., the Paasche final demand deflator will equal to the national Paasche value added deflator. To see why this is so, use the Paasche formula in (133) and for the left hand side index, collect all of the quantity terms both in the numerator and denominator of the index that correspond to the common establishment price for the nth commodity, $p_n^t = p_{yn}^{et} = p_{xn}^{et}$, for e = 1,...,E. Using (131) for t =1, the resulting sum of collected quantity terms will sum to f_n^1. Since this is true for n = 1,...,N, it can be seen that the left hand side Paasche index is equal to the right hand side Paasche index.

The results in the previous two paragraphs imply that the national value added deflator will equal the final demand deflator provided that Paasche or Laspeyres indexes are used and provided that assumptions (134) hold. But these two results immediately imply that if (134) holds and Fisher ideal price indexes are used, then an important equality is obtained: *the national value added deflator equals the final demand deflator*.

Define the *national share*[73] *of value added* for establishment e and netput n for period t as follows:

$$(135)\quad s_n^{et} \equiv p_n^{et} q_n^{et} / \sum_{i=1}^{E} \sum_{j=1}^{2N} p_j^{it} q_j^{it} \,; \qquad n = 1,...,2N \,;\ e = 1,...,E \,;\ t = 0,1$$

[72] Under these hypotheses, the vector of producer prices p^t can be interpreted as the vector of *basic producer prices* that appears in the System of National Accounts, 1993.

[73] These shares sum to unity but they are not “normal” nonnegative shares that sum to one since the establishment shares that correspond to intermediate inputs are negative (or 0).

$= p_n^{et}q_n^{et}/va^t$

where $va^t \equiv \sum_{i=1}^{E} \sum_{j=1}^{2N} p_j^{it}q_j^{it}$ is the national value added for period t. Using the above establishment value added shares and the establishment output and intermediate input price relatives, p_n^{e1}/p_n^{e0}, we can define the logarithm of the *national Törnqvist* (1936)(1937) *Theil* (1967) *value added deflator* P_T as follows:

(136) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{e=1}^{E} \sum_{n=1}^{2N} (1/2)(s_n^{e0} + s_n^{e1}) \ln (p_n^{e1}/p_n^{e0})$.

Now make the price equality assumptions (134) and start with the national Törnqvist Theil value added deflator and collect all of the exponents that correspond to the common price relative for commodity n, p_n^1/p_n^0. The sum of these exponents is:

$$(137)\ (1/2)[\textstyle\sum_{e=1}^{E} p_{yn}^{e0}y_n^{e0} - \sum_{e=1}^{E} p_{xn}^{e0}x_n^{e0}]/va^0 + (1/2)[\sum_{e=1}^{E} p_{yn}^{e1}y_n^{e1} - \sum_{e=1}^{E} p_{xn}^{e1}x_n^{e1}]/va^1$$
$$= (1/2)[\textstyle\sum_{e=1}^{E} p_n^{0}y_n^{e0} - \sum_{e=1}^{E} p_n^{0}x_n^{e0}]/va^0 + (1/2)[\sum_{e=1}^{E} p_n^{1}y_n^{e1} - \sum_{e=1}^{E} p_n^{1}x_n^{e1}]/va^1 \quad \text{using (134)}$$
$$= (1/2)p_n^0f_n^0/va^0 + (1/2)p_n^1f_n^1/va^1 \quad \text{using (132).}$$

But the right hand side of (137) is the exponent for the nth price term, $p_{fn}^1/p_{fn}^0 = p_n^1/p_n^0$ in the Törnqvist Theil final demand deflator. Since this equality holds for all n = 1,...,N, the equality of the national value added deflator to the final demand deflator is also obtained if the Törnqvist formula P_T is used on both sides of (133); i.e., we have shown that under assumptions (132) (unit value prices for final outputs) and (134) (each establishment faces the same prices for outputs and intermediate inputs), then

(138) $P_T(p_f^0,p_f^1,f^0,f^1) = P_T(p^0,p^1,q^0,q^1)$.

Summarizing the above results, we have shown that the national value added deflator is equal to the final demand deflator provided that all establishments face the same vector of prices in each period for both the outputs that they produce and for the intermediate inputs that they use and provided that either the Laspeyres, Paasche, Fisher or Törnqvist price index formula is used for both deflators.[74] However, these results were established ignoring the existence of indirect taxes and subsidies that may be applied to the outputs and intermediate inputs of each establishment. It is necessary to extend the initial results to deal with situations where there are indirect taxes on deliveries to final demand and indirect taxes on the use of intermediate inputs.

Again, it is assumed that all establishments face the same prices for their inputs and outputs but it is now assumed that their deliveries to the final demand sector are *taxed*.[75]

[74] This result does not carry over if we use the Walsh price index formula.

[75] Hicks (1940; 106) appears to have been the first to note that the treatment of indirect taxes in national income accounting depends on the purpose for which the calculation is to be used. Thus for measuring productivity, Hicks (1940; 124) advocated using prices that best represented marginal costs and benefits from the perspective of producers; i.e., basic prices should be used. On the other hand, if the measurement of economic welfare is required, Hicks (1940; 123-124) advocated the use of prices that best represent marginal utilities of consumers; i.e., final demand prices should be used. Bowley (1922; 8) advocated the use of final demand prices but he implicitly took a welfare point of view: "To the purchaser of whisky,

Let τ_n^t be the period t ad valorem commodity tax rate on deliveries to final demand of commodity n for t = 0,1 and n = 1,...,N.[76] Thus the period t final demand price for commodity n is now:

$$(139) \quad p_{fn}^t = p_n^t(1+\tau_n^t) \; ; \qquad n = 1,...,N \; ; \; t = 0,1.$$

The tax adjusted final demand prices defined by (139) can be used to form new vectors of final demand price vectors, $p_f^t \equiv [p_{f1}^t,...,p_{fN}^t]$ for t = 0,1. The corresponding final demand quantity vectors, f^0 and f^1, are still defined by the commodity balance equations, (131). Now pick an index number formula P and form the *final demand deflator* $P(p_f^0,p_f^1,f^0,f^1)$ using the new tax adjusted prices, p_f^0,p_f^1. Obviously, if the commodity tax rates τ_n^t are substantial, the new final demand deflator $P(p_f^0,p_f^1,f^0,f^1)$ can be substantially different from the national value added deflator $P(p^0,p^1,q^0,q^1)$ defined earlier in this section (because all of the commodity tax terms are missing from the national value added deflator).

However, it is possible to adjust our old national value added deflator in an attempt to make it more comparable to the final demand deflator. Recall that the price and quantity vectors, p^t and q^t, that appear in the national value added deflator are defined as follows:[77]

$$(140) \quad \mathbf{p^t} \equiv [p_y^{1t},p_x^{1t};p_y^{2t},p_x^{2t};\ldots;p_y^{Et},p_x^{Et}] \; ; \qquad t = 0,1;$$
$$\mathbf{q^t} \equiv [y^{1t},-x^{1t};y^{2t},-x^{2t};...;y^{Et},-x^{Et}] \; ; \qquad t = 0,1;$$

where p_y^{et} is the vector of output prices that establishment e faces in period t, p_x^{et} is the vector of input prices that establishment e faces in period t, y^{et} is the production vector for establishment e in period t and x^{et} is the vector of intermediate inputs used by establishment e during period t. The adjustment that is made to the national value added deflator is that an additional N artificial commodities is added to the list of outputs and inputs that the national value added deflator aggregates over. Define the price and quantity of the nth extra *artificial commodity* as follows:[78]

$$(141) \quad p_n^{At} \equiv p_n^t\tau_n^t \; ; \; q_n^{At} \equiv f_n^t \; ; \qquad n = 1,...,N \; ; \; t = 0,1.$$

tobacco and entertainment tickets, the goods bought are worth what he pays; it is indifferent to him whether the State or the producer gets the money.”

[76] If commodity n is subsidized during period t, then τ_n^t can be set equal to minus the subsidy rate. In most countries, the commodity tax regime is much more complex than we have modeled it above, in that some sectors of final demand are taxed differently than other sectors; e.g., exported commodities are generally not taxed or are taxed more lightly than other final demand sectors. To deal with these complications, it would be necessary to decompose the single final demand sector into a number of sectors (e. g., the familiar C + I + G + X – M decomposition) where the tax treatment in each sector was uniform. In this disaggregated framework, tariffs on imported goods and services can readily be accommodated. There are additional complications due to the existence of commodity taxes that fall on intermediate inputs. To deal adequately with all of these complications would require a rather extended discussion. Our purpose here is to indicate to the reader that the national value added deflator is closely connected to the final demand deflator.

[77] Of course, under assumptions (134), the definition of p^t simplifies dramatically.

[78] Again we are assuming that (134) holds; i.e., that each establishment faces the same prices for each intermediate input and output.

Thus the period t price of the nth artificial commodity is just the product of the nth basic price, p_n^t, times the nth commodity tax rate in period t, τ_n^t. The period t quantity for the nth artificial commodity is simply equal to period t final demand for commodity n, f_n^t. Note that the period t value of all N artificial commodities is just equal to period t commodity tax revenue. Define the period t price and quantity vectors for the artificial commodities in the usual way; i.e., $p^{At} \equiv [p_1^{At},...,p_N^{At}]$ and $q^{At} \equiv [q_1^{At},...,q_N^{At}] = f^t$, t = 0,1. Now add the extra price vector p^{At} to our old period t price vector $\mathbf{p}^{\mathbf{t}}$ that was used in the national value added deflator and add the extra quantity vector q^{At} to our old period t quantity vector $\mathbf{q}^{\mathbf{t}}$ that was used in the national value added deflator; i.e., define the *augmented national price and quantity vectors*, p^{t*} and q^{t*} as follows:

$$(142)\quad p^{t*} \equiv [\mathbf{p}^{\mathbf{t}},p^{At}] \; ; \; q^{t*} \equiv [\mathbf{q}^{\mathbf{t}},q^{At}] \; ; \qquad\qquad t = 0,1.$$

Using the augmented price and quantity vectors defined above, calculate a *new tax adjusted national value added deflator* using the chosen index number formula, $P(p^{0*},p^{1*},q^{0*},q^{1*})$, and ask whether it will equal the final demand deflator, $P(p_f^0,p_f^1,f^0,f^1)$ using the new tax adjusted prices, p_f^0,p_f^1, defined by (139); i.e., we ask whether the following equality holds:

$$(143)\quad P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p_f^0,p_f^1,f^0,f^1).$$

Choose P to be P_L, the Laspeyres formula and evaluate the left hand side of (143). The numerator of this Laspeyres value added index is:

$$\begin{aligned}(144)\ & \textstyle\sum_{i=1}^{E} \sum_{j=1}^{2N} p_j^{i1}q_j^{i0} + \sum_{n=1}^{N} p_n^{A1}q_n^{A0} \\ &= \textstyle\sum_{n=1}^{N} \sum_{e=1}^{E} p_n^1 y_n^{e0} - \sum_{n=1}^{N} \sum_{e=1}^{E} p_n^1 x_n^{e0} + \sum_{n=1}^{N} p_n^1 \tau_n^1 f_n^0 && \text{using (134) and (141)} \\ &= \textstyle\sum_{n=1}^{N} p_n^1\{\sum_{e=1}^{E} y_n^{e0} - \sum_{e=1}^{E} x_n^{e0}\} + \sum_{n=1}^{N} p_n^1 \tau_n^1 f_n^0 \\ &= \textstyle\sum_{n=1}^{N} p_n^1\{f_n^0\} + \sum_{n=1}^{N} p_n^1 \tau_n^1 f_n^0 && \text{using (131) for } t = 0 \\ &= \textstyle\sum_{n=1}^{N} p_n^1(1 + \tau_n^1)f_n^0.\end{aligned}$$

Thus the sum of the terms in the numerator of the national tax adjusted Laspeyres value added deflator, $P_L(p^{0*},p^{1*},q^{0*},q^{1*})$, involving p_n^1 is $p_n^1(1+\tau_n^1)f_n^0$, which is equal to the nth term in the numerator of the Laspeyres final demand deflator, $P_L(p_f^0,p_f^1,f^0,f^1)$. In a similar fashion, collect up all terms in the denominator of the Laspeyres national value added deflator $P_L(p^{0*},p^{1*},q^{0*},q^{1*})$ that correspond to the nth commodity price p_n^0. Using (131) for t = 0, it is found that the sum of these terms involving p_n^0 is $p_n^0(1+\tau_n^1)f_n^0$ which is equal to the nth term in the denominator of the final demand deflator, $P_L(p_f^0,p_f^1,f^0,f^1)$. Thus (143) does hold as an exact equality under our assumptions if the Laspeyres price index is used for each of the deflators.

Now choose P to be P_P, the Paasche formula and evaluate the left hand side of (143). The denominator of this Paasche value added index is:

$$\begin{aligned}(145)\ & \textstyle\sum_{i=1}^{E} \sum_{j=1}^{2N} p_j^{i0}q_j^{i1} + \sum_{n=1}^{N} p_n^{A0}q_n^{A1} \\ &= \textstyle\sum_{n=1}^{N} \sum_{e=1}^{E} p_n^0 y_n^{e1} - \sum_{n=1}^{N} \sum_{e=1}^{E} p_n^0 x_n^{e1} + \sum_{n=1}^{N} p_n^0 \tau_n^0 f_n^1 && \text{using (134) and (141)}\end{aligned}$$

$$= \sum_{n=1}^{N} p_n^{0}\{\sum_{e=1}^{E} y_n^{e1} - \sum_{e=1}^{E} x_n^{e1}\} + \sum_{n=1}^{N} p_n^{0}\tau_n^{0}f_n^{1}$$
$$= \sum_{n=1}^{N} p_n^{0}\{f_n^{1}\} + \sum_{n=1}^{N} p_n^{0}\tau_n^{0}f_n^{1} \qquad \text{using (131) for } t = 1$$
$$= \sum_{n=1}^{N} p_n^{0}(1 + \tau_n^{0})f_n^{1}.$$

Thus the sum of the terms in the denominator of the national tax adjusted Paasche value added deflator, $P_P(p^{0*},p^{1*},q^{0*},q^{1*})$, involving p_n^0 is $p_n^0(1+\tau_n^0)f_n^1$, which is equal to the nth term in the denominator of the Paasche final demand deflator, $P_P(p_f^0,p_f^1,f^0,f^1)$. In a similar fashion, collect up all terms in the numerator of the Paasche national value added deflator $P_P(p^{0*},p^{1*},q^{0*},q^{1*})$ that correspond to the nth commodity price p_n^1. Using (131) for t = 1, it is found that the sum of these terms involving p_n^1 is $p_n^1(1+\tau_n^1)f_n^1$ which is equal to the nth term in the numerator of the Paasche final demand deflator, $P_P(p_f^0,p_f^1,f^0,f^1)$. Thus (143) does hold as an exact equality under our assumptions if the Paasche price index is used for each of the deflators.

Finally, choose P to be P_T, the Törnqvist Theil formula for a price index, and evaluate both sides of (143). In general, this time an exact equality is *not* obtained between the national Törnqvist Theil tax adjusted value added deflator $P_T(p^{0*},p^{1*},q^{0*},q^{1*})$ and the Törnqvist Theil final demand deflator $P_T(p_f^0,p_f^1,f^0,f^1)$. To see why this is so, define the *national share of value added* for establishment e and netput n for period t as follows:

$$(146)\quad s_n^{et} \equiv p_n^{et}q_n^{et}/[\sum_{i=1}^{E}\sum_{j=1}^{2N} p_j^{it}q_j^{it} + \sum_{i=1}^{N} p_i^{At}q_i^{At}]\ ; \quad n = 1,\ldots,2N\ ;\ e = 1,\ldots,E\ ;\ t = 0,1$$
$$= p_n^{et}q_n^{et}/tva^t$$

where $tva^t \equiv \sum_{i=1}^{E}\sum_{j=1}^{2N} p_j^{it}q_j^{it} + \sum_{n=1}^{N} p_n^{At}q_n^{At}$ is the tax augmented national value added for period t. We also need to define the shares for the N artificial commodities as follows:

$$(147)\quad s_n^{At} \equiv p_n^{At}q_n^{At}/[\sum_{i=1}^{E}\sum_{j=1}^{2N} p_j^{it}q_j^{it} + \sum_{i=1}^{N} p_i^{At}q_i^{At}]\ ; \qquad n = 1,\ldots,N\ ;\ t = 0,1$$
$$= p_n^{t}\tau_n^{t}f_n^{t}/tva^t \qquad \text{using (141).}$$

Using the above establishment value added shares and the establishment output and intermediate input price relatives, p_n^{e1}/p_n^{e0}, we can define the logarithm of the *national Törnqvist* (1936)(1937) *Theil* (1967) *tax augmented value added deflator* P_T as follows:

$$(148)\quad \ln P_T(p^{0*},p^{1*},q^{0*},q^{1*})$$
$$\equiv \sum_{e=1}^{E}\sum_{n=1}^{2N} (1/2)(s_n^{e0} + s_n^{e1}) \ln (p_n^{e1}/p_n^{e0}) + \sum_{n=1}^{N} (1/2)(s_n^{A0} + s_n^{A1}) \ln (p_n^{A1}/p_n^{A0})$$
$$= \sum_{e=1}^{E}\sum_{n=1}^{2N} (1/2)(s_n^{e0} + s_n^{e1}) \ln (p_n^{e1}/p_n^{e0}) + \sum_{n=1}^{N} (1/2)(s_n^{A0} + s_n^{A1}) \ln (p_n^{1}\tau_n^{1}/p_n^{0}\tau_n^{0})$$

where the last equality follows using (141). Now make the price equality assumptions (134) and start with the national tax augmented Törnqvist Theil value added deflator and collect all of the exponents that correspond to the common price relative for commodity n, p_n^1/p_n^0. The sum of these commodity n exponents plus the exponents that correspond to the nth artificial price relative p_n^{A1}/p_n^{A0} is:

$$(149)\quad (1/2)[\sum_{e=1}^{E} p_{yn}^{e0}y_n^{e0} - \sum_{e=1}^{E} p_{xn}^{e0}x_n^{e0}]/tva^0$$
$$+ (1/2)[\sum_{e=1}^{E} p_{yn}^{e1}y_n^{e1} - \sum_{e=1}^{E} p_{xn}^{e1}x_n^{e1}]/tva^1 + (1/2)[p_n^{A0}q_n^{A0}]/tva^0 + (1/2)[p_n^{A1}q_n^{A1}]/tva^1$$

$$= (1/2)[\sum_{e=1}^{E} p_n^0 y_n^{e0} - \sum_{e=1}^{E} p_n^0 x_n^{e0}]/tva^0 + (1/2)[\sum_{e=1}^{E} p_n^1 y_n^{e1} - \sum_{e=1}^{E} p_n^1 x_n^{e1}]/tva^1$$
$$+ (1/2)[p_n^0 \tau_n^0 f_n^0]/tva^0 + (1/2)[p_n^1 \tau_n^1 f_n^1]/tva^1 \qquad \text{using (134) and (141)}$$
$$= (1/2)p_n^0 f_n^0/tva^0 + (1/2)p_n^1 f_n^1/tva^1 + (1/2)[p_n^0 \tau_n^0 f_n^0]/tva^0 + (1/2)[p_n^1 \tau_n^1 f_n^1]/tva^1 \qquad \text{using (132)}$$
$$= (1/2)p_n^0(1+\tau_n^0)f_n^0/tva^0 + (1/2)p_n^1(1+\tau_n^1)f_n^1/tva^1.$$

Substituting (149) into (148), we obtain the following expression for the logarithm of the Törnqvist Theil tax augmented value added deflator:

$$(150)\ \ln P_T(p^0*,p^1*,q^0*,q^1*)$$
$$= (1/2)\sum_{n=1}^{N} [\{p_n^0(1+\tau_n^0)f_n^0/tva^0\} + \{p_n^1(1+\tau_n^1)f_n^1/tva^1\}] \ln (p_n^1/p_n^0)$$
$$+ \sum_{n=1}^{N} (1/2)(s_n^{A0} + s_n^{A1}) \ln (\tau_n^1/\tau_n^0).$$

On the other hand, the logarithm of the *Törnqvist Theil final demand deflator* is defined as follows:

$$(151)\quad \ln P_T(p_f^0,p_f^1,f^0,f^1) \equiv \sum_{n=1}^{N} (1/2)(s_{nf}^0 + s_{nf}^1) \ln (p_{nf}^1/p_n^{f0})$$
$$= \sum_{n=1}^{N} (1/2)(s_{nf}^0 + s_{nf}^1) \ln [p_n^1(1+\tau_n^1)/p_n^0(1+\tau_n^0)]$$
$$= \sum_{n=1}^{N} (1/2)[\{p_n^0(1+\tau_n^0)f_n^0/tva^0\} + \{p_n^1(1+\tau_n^1)f_n^1/tva^1\}]\ln [p_n^1(1+\tau_n^1)/p_n^0(1+\tau_n^0)]$$
$$= \sum_{n=1}^{N} (1/2)[\{p_n^0(1+\tau_n^0)f_n^0/tva^0\} + \{p_n^1(1+\tau_n^1)f_n^1/tva^1\}]\ln (p_n^1/p_n^0)$$
$$+ \sum_{n=1}^{N} (1/2)(s_{nf}^0 + s_{nf}^1) \ln [(1+\tau_n^1)/(1+\tau_n^0)].$$

Comparing (150) and (151), it can be seen that in general,

$$(152)\ P_T(p^0*,p^1*,q^0*,q^1*) \neq P_T(p_f^0,p_f^1,f^0,f^1).$$

However, if the extra assumption is made that the commodity tax rates are equal in periods 0 and 1 so that

$$(153)\quad \tau_n^0 = \tau_n^1 \qquad \text{for } n = 1,\dots,N,$$

then it can be seen that the national Törnqvist Theil tax adjusted value added deflator $P_T(p^0*,p^1*,q^0*,q^1*)$ and the Törnqvist Theil final demand deflator $P_T(p_f^0,p_f^1,f^0,f^1)$ are *exactly equal*.

The last few results can be modified to work in reverse; i.e., start with the final demand deflator, make some adjustments to it using artificial commodities, and then the resulting tax adjusted final demand deflator can equal the original unadjusted national value added deflator. To implement this reverse procedure, it is necessary to add an additional N artificial commodities to the list of outputs and inputs that the final demand deflator aggregates over. Define the price and quantity of the nth extra *artificial commodity* as follows:

$$(154)\quad p_n^{At} \equiv p_n^t \tau_n^t\ ;\ q_n^{At} \equiv -f_n^t\ ; \qquad n = 1,\dots,N\ ;\ t = 0,1.$$

Thus the period t price of the nth artificial commodity is just the product of the nth basic price, p_n^t, times the nth commodity tax rate in period t, τ_n^t. The period t quantity for the nth artificial commodity is simply equal to minus period t final demand for commodity n, $-f_n^t$. Note that the period t value of all N artificial commodities is just equal to *minus* period t commodity tax revenue. Define the period t price and quantity vectors for the artificial commodities in the usual way; i.e., $p^{At} \equiv [p_1^{At},...,p_N^{At}]$ and $q^{At} \equiv [q_1^{At},...,q_N^{At}] = f^t$, t = 0,1. The extra price vector p^{At} is now added to our old period t price vector p_f^t that was used in the final demand deflator and the extra quantity vector q^{At} is added to our old period t quantity vector f^t that was used in the final demand deflator; i.e., define the *augmented final demand price and quantity vectors*, $p_f^{t}*$ and $f^{t}*$ as follows:

$$(155)\quad p_f^{t}* \equiv [p_f^t,p^{At}]\ ;\ f^{t}* \equiv [f^t,q^{At}]\ ; \qquad\qquad t = 0,1.$$

Using the augmented price and quantity vectors defined above, calculate a *new tax adjusted final demand deflator* using the chosen index number formula, $P(p_f^{0}*,p_f^{1}*,f^{0}*,f^{1}*)$, and we ask whether it will equal our initial *national value added deflator* (that did not make any tax adjustments for commodity taxes on final demands), $P(p^0,p^1,q^0,q^1)$; i.e., whether the following equality holds:

$$(156)\quad P(p_f^{0}*,p_f^{1}*,f^{0}*,f^{1}*) = P(p^0,p^1,q^0,q^1).$$

Under the assumption that all establishments face the same prices, it can be shown *that the tax adjusted final demand deflator will exactly equal the national value added deflator* provided that the index number formula in (156) is chosen to be the Laspeyres, Paasche or Fisher formulae, P_L, P_P or P_F. In general, (156) will not hold as an exact equality if the Törnqvist Theil formula P_T is used. However, if the commodity tax rates are equal in periods 0 and 1, so that assumptions (153) hold in addition to assumptions (134), then it can be shown that (156) will hold as an exact equality when P is set equal to P_T, the Törnqvist Theil formula. These results are of some practical importance for the following reason. Most countries do not have adequate surveys that will support a complete system of value added price indexes for each sector of the economy.[79] Adequate information is generally available that will enable the statistical agency to calculate the final demand deflator. However, for measuring the productivity of the economy using the economic approach to index number theory, the national value added deflator is the preferred deflator.[80] The results in this section show how the final demand deflator can be modified to give a close approximation to the national value added deflator under certain conditions.

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[79] In particular, information on the prices and quantities of intermediate inputs used by sector are generally lacking. These data deficiencies were noted by Fabricant (1938; 566-570) many years ago and he indicated some useful methods that are still used today in attempts to overcome these data deficiencies.

[80] See Schreyer (2001) for more explanation. Basically, in order to evaluate the efficiency of the production sector, we should use prices that producers actually face. For inputs, this means that after tax prices should be used and for outputs, before commodity tax prices should be used. This point was emphasized by Jorgenson and Griliches (1967).

It has always been a bit of a mystery how tax payments should be decomposed into price and quantity components in national accounting theory. The results presented in this section may be helpful in suggesting "reasonable" decompositions under certain conditions.

19. Midyear Indexes as Approximations to Superlative Indexes

Recall definitions (19) and (20) in section chapter 3, which defined the Marshall (1887) Edgeworth (1925), $P_{ME}(p^0,p^1,q^0,q^1)$, and Walsh (1901; 398) (1921; 97), $P_W(p^0,p^1,q^0,q^1)$, price indexes between periods 0 and 1 respectively. In section 9 above, we indicated that the Walsh price index is a superlative index. On the other hand, although the Marshall Edgeworth price index is not superlative, Diewert (1978; 897) showed that it will approximate any superlative index to the second order around a point where the base and current period price and quantity vectors are equal[81] so that usually, P_{ME} will approximate a superlative index fairly closely. In this section, we will draw on some recent results due to Schultz (1999) and Okamoto (2001) and show how various *midyear price indexes* can approximate Walsh or Marshall Edgeworth indexes fairly closely under certain conditions. As we shall see, midyear indexes do not rely on quantity weights for the current and base periods; rather they utilize quantity weights from years that lie between the base period and current period and hence, they can be produced on a timely basis.

Let t be an even positive integer. Then Schultz (1999) defined a *midyear price index*, which compares the price vector in period t, p^t, to the corresponding price vector in period 0, p^0, as follows:

$$(157)\quad P_S(p^0,p^t,q^{t/2}) \equiv \sum_{n=1}^N p_n^t q_n^{t/2} / \sum_{n=1}^N p_n^0 q_n^{t/2}$$

where $q^{t/2}$ is the quantity vector that pertains to the intermediate period, t/2. The definition for a midyear price index when t is odd (and greater than 2) is a bit trickier. Okamoto (2001) defined *arithmetic type* and *geometric type midyear price indexes* comparing prices in period 0 with period t where t is odd by (158) and (159) respectively:

$$(158)\quad P_{OA}(p^0,p^t,q^{(t-1)/2},q^{(t+1)/2}) \equiv \sum_{n=1}^N p_n^t(1/2)(q_n^{(t-1)/2}+q_n^{(t+1)/2}) / \sum_{n=1}^N p_n^0(1/2)(q_n^{(t-1)/2}+q_n^{(t+1)/2})$$

$$(159)\quad P_{OG}(p^0,p^t,q^{(t-1)/2},q^{(t+1)/2}) \equiv \sum_{n=1}^N p_n^t(q_n^{(t-1)/2}q_n^{(t+1)/2})^{1/2} / \sum_{n=1}^N p_n^0(q_n^{(t-1)/2}q_n^{(t+1)/2})^{1/2}.$$

Each of the price indexes defined by (158) and (159) is of the fixed basket type. In the arithmetic type index defined by (158), the fixed basket quantity vector is the simple arithmetic average of the two quantity vectors that pertain to the intermediate periods, (t−1)/2 and (t+1)/2, whereas in the geometric type index defined by (159), the reference quantity vector is the geometric average of these two intermediate period quantity vectors.

[81] This result can be generalized to points of approximation where $p^1 = \alpha p^0$ and $q^1 = \beta q^0$; i.e., points where the period 1 price vector is proportional to the period 0 price vector and where the period 1 quantity vector is proportional to the period 0 quantity vector.

Okamoto (2001) used the above definitions in order to define the following sequence of *fixed base (arithmetic type) midyear price indexes*:

(160) $1, P_{ME}(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_{OA}(p^0,p^3,q^1,q^2), P_S(p^0,p^4,q^2), P_{OA}(p^0,p^5,q^2,q^3), ...$.

Thus in period 0, the index is set equal to 1. In period 1, the index is set equal to the Marshall Edgeworth price index between periods 0 and 1, $P_{ME}(p^0,p^1,q^0,q^1)$, (which is the only index number in the above sequence that requires information on current period quantities). In period 2, the index is set equal to the Schultz midyear index, $P_S(p^0,p^2,q^1)$, defined by (157), which uses the quantity weights of the prior period 1, q^1. In period 3, the index is set equal to the arithmetic Okamoto midyear index, $P_{OA}(p^0,p^3,q^1,q^2)$, defined by (158), which uses the quantity weights of the two prior periods, q^1 and q^2. And so on.

Okamoto (2001) also used the above definitions in order to define the following sequence of *fixed base (geometric type) midyear price indexes*:

(161) $1, P_W(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_{OG}(p^0,p^3,q^1,q^2), P_S(p^0,p^4,q^2), P_{OG}(p^0,p^5,q^2,q^3), ...$.

Thus in period 0, the index is set equal to 1. In period 1, the index is set equal to the Walsh price index between periods 0 and 1, $P_W(p^0,p^1,q^0,q^1)$, (which is the only index number in the sequence that requires information on current period quantities). In period 2, the index is set equal to the Schultz midyear index, $P_S(p^0,p^2,q^1)$. In period 3, the index is set equal to the (geometric type) Okamoto midyear index, $P_{OG}(p^0,p^3,q^1,q^2)$, defined by (159), which uses the quantity weights of the two prior periods, q^1 and q^2 and so on.

It is also possible to define *chained* sequences[82] of midyear indexes that are counterparts to the fixed base sequences defined by (160) and (161). Thus a chained counterpart to (160) can be defined as follows:

(162) $1,P_{ME}(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_{ME}(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3),$
$P_{ME}(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2)P_S(p^3,p^5,q^4), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3)P_S(p^4,p^6,q^5), ...$.

A chained counterpart to (161) can be defined as follows:

(163) $1, P_W(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_W(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3),$
$P_W(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2)P_S(p^3,p^5,q^4), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3)P_S(p^4,p^6,q^5), ...$.

Note that (162) and (163) differ only in the use of the Marshall Edgeworth index, $P_{ME}(p^0,p^1,q^0,q^1)$, to compare prices in period 1 to period 0, versus the Walsh index, $P_W(p^0,p^1,q^0,q^1)$, which is also used to compare prices for the same two periods. Otherwise, only the basic Schultz midyear formula, $P_S(p^t,p^{t+2},q^{t+1})$ is used in both (162) and (163).

[82] See chapter 8 for a review of chained indexes.

Schultz (1999) and Okamoto (2001) showed, using Canadian and Japanese data, that midyear index number sequences like those defined by (160) and (161) above are reasonably close to their superlative Fisher ideal counterparts.

In addition to the above empirical results, we can generate some theoretical results that support the use of midyear indexes as approximations to superlative indexes.[83] Our theoretical results presented below rely on specific assumptions about how the quantity vectors q^t change over time. We will make two such specific assumptions.

Thus we now assume that there are *linear trends in quantities* over our sample period; i.e., we assume that:

(164) $q^t = q^0 + t\alpha$; $\quad t = 1,...,T$

where $\alpha \equiv [\alpha_1,..., \alpha_N]$ is a vector of constants. Hence for t even, we have, using (164), that:

(165) $(1/2)q^0 + (1/2)q^t = (1/2)q^0 + (1/2)[q^0 + t\alpha] = q^0 + (t/2)\alpha = q^{t/2}$.

Similarly, for t odd (and greater than 2), we have:

$$
\begin{aligned}
(166)\quad (1/2)q^0 + (1/2)q^t &= (1/2)q^0 + (1/2)[q^0 + t\alpha] \\
&= (1/2)q^0 + (1/2)[q^0 + \{(1/2)(t-1)+(1/2)(t+1)\}\alpha] \\
&= (1/2)[q^0 +\{(t-1)/2\}\alpha] + (1/2)[q^0 +\{(t+1)/2\}\alpha] \\
&= (1/2)q^{(t-1)/2} + (1/2)\, q^{(t+1)/2}.
\end{aligned}
$$

Thus under the linear time trends in quantities assumption (164), we can show, using (165) and (166), that the Schultz midyear and the Okamoto arithmetic type midyear indexes all equal their Marshall Edgeworth counterparts; i.e., we have:

(167) $P_S(p^0,p^t,q^{t/2}) = P_{ME}(p^0,p^t,q^0,q^t)$ $\quad$ for t even;
(168) $P_{OA}(p^0,p^t,q^{(q-1)/2},q^{(q+1)/2}) = P_{ME}(p^0,p^t,q^0,q^t)$ $\quad$ for t odd.

Thus under the linear trends assumption (164), the fixed base and chained arithmetic type sequences of midyear indexes, (160) and (162) respectively, become the following sequences of Marshall Edgeworth indexes[84]:

(169) $1, P_{ME}(p^0,p^1,q^0,q^1), P_{ME}(p^0,p^2,q^0,q^2), P_{ME}(p^0,p^3,q^0,q^3), P_{ME}(p^0,p^4,q^0,q^4), ...$;
(170) $1, P_{ME}(p^0,p^1,q^0,q^1), P_{ME}(p^0,p^2,q^0,q^2), P_{ME}(p^0,p^1,q^0,q^1)P_{ME}(p^1,p^3,q^1,q^3),$
$P_{ME}(p^0,p^2,q^0,q^2)P_{ME}(p^2,p^4,q^2,q^4), P_{ME}(p^0,p^1,q^0,q^1)P_{ME}(p^1,p^3,q^1,q^3)P_{ME}(p^3,p^5,q^3,q^5),$
... .

[83] Okamoto (2001) also makes some theoretical arguments relying on the theory of Divisia indexes to show why midyear indexes might approximate superlative indexes.
[84] Recall that Marshall Edgeworth indices are not actually superlative but they will usually approximate their superlative Fisher counterparts fairly closely using "normal" time series data.

For our second specific assumption about the behavior of quantities over time, we assume that quantities are growing at *geometric rates* over the sample period; i.e., we assume that:

(171) $q_n^t = (1+g_n)^t q_n^0$; $n = 1,...,N$; $t = 1,...,T$

where g_n is the geometric growth rate for quantity n. Hence for t even, using (171), we have:

(172) $[q_n^0 q_n^t]^{1/2} = (1+g_n)^{t/2} q_n^0 = q_n^{t/2}$.

For t odd (and greater than 2), again using (171), we have:

(173) $[q_n^0 q_n^t]^{1/2} = (1+g_n)^{t/2} q_n^0 = (1+g_n)^{(1/4)[(t-1)+(t+1)]} q_n^0 = [q_n^{(t-1)/2} q_n^{(t+1)/2}]^{1/2}$.

Using (172) and (173), we can show that if quantities grow geometrically, then the Schultz midyear and the Okamoto geometric type midyear indexes all equal their Walsh counterparts; i.e., we have:

(174) $P_S(p^0,p^t,q^{t/2}) = P_W(p^0,p^t,q^0,q^t)$ for t even;
(175) $P_{OG}(p^0,p^t,q^{(q-1)/2}, q^{(q+1)/2}) = P_W(p^0,p^t,q^0,q^t)$ for t odd.

Thus under the geometric growth rates assumption (171), the fixed base and chained geometric type sequences of midyear indexes, (161) and (163) respectively, become the following sequences of Walsh price indices:

(176) $1, P_W(p^0,p^1,q^0,q^1), P_W(p^0,p^2,q^0,q^2), P_W(p^0,p^3,q^0,q^3), P_W(p^0,p^4,q^0,q^4), ...$;
(177) $1, P_W(p^0,p^1,q^0,q^1), P_W(p^0,p^2,q^0,q^2), P_W(p^0,p^1,q^0,q^1)P_W(p^1,p^3,q^1,q^3),$
$P_W(p^0,p^2,q^0,q^2)P_W(p^2,p^4,q^2,q^4), P_W(p^0,p^1,q^0,q^1)P_W(p^1,p^3,q^1,q^3)P_W(p^3,p^5,q^3,q^5), ...$.

Since the Walsh price indexes are superlative, the results in this section show that if quantities are trending in a very smooth manner, then it is likely that we can approximate superlative indexes fairly closely without having a knowledge of current period quantities (but provided that lagged quantity vectors can be estimated on a continuous basis).

It seems very likely that the above midyear indexes will approximate superlative indexes to a much higher degree of approximation than chained or fixed base Laspeyres indexes.[85] However, the real choice may not be between computing Laspeyres indexes versus midyear indexes but in producing midyear indexes in a timely manner versus waiting a year or two to produce actual superlative indexes. However, there is always the danger that when price or quantity trends suddenly change, the midyear indexes considered above could give rather misleading advanced estimates of a superlative index. However, if this limitation of midyear indexes is kept in mind, it seems that it would be

[85] It is clear that the midyear index methodologies could be regarded as very simple forecasting schemes to *estimate* the current period quantity vector based on past time series of quantity vectors. Viewed in this way, these midyear methods could be greatly generalized using time series forecasting methods.

generally useful for statistical agencies to compute midyear indexes on an experimental basis.[86] In the next chapter, we will compare midyear indexes with their superlative counterparts.

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[86] Okamoto (2001) notes that in the 2000 Japanese CPI revision, midyear indexes and chained Laspeyres indexes will be added as a set of supplementary indexes to the usual fixed base Laspeyres price index.

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